Section 3.4 **Derivatives of trigonometric functions**

Theorem. $\lim_{\theta \to 0} \sin\theta = 0.$

Theorem. $\lim_{\theta \to 0} \cos\theta = 1.$

Theorem. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ $\quad \lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1 \quad \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1 \quad \lim_{\theta \to 0} \frac{\theta}{\tan\theta} = 1$

Corollary. $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0.$

$$\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \frac{\frac{\sin\theta}{\cos\theta}}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta} = \frac{\lim_{\theta \to 0} \frac{\sin\theta}{\theta}}{\lim_{\theta \to 0} \cos\theta}$$

$$= \frac{1}{1} = 1$$

Example 1. Find each limit

(a.) $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{2x^2}$

$1-\cos 2x = 2\sin^2 x$

$$= \lim_{x\to 0} \frac{\cancel{2}\sin^2 x}{\cancel{2}x^2} = \lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = \left[\lim_{x\to 0} \frac{\sin x}{x}\right]^2 = \boxed{1}$$

$$\boxed{\lim_{☺\to 0} \frac{\sin(☺)}{☺} = 1}$$

(b.) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{2x} = \dfrac{1}{2}\lim\limits_{x\to 0} \dfrac{5\sin 5x}{5x} = \dfrac{5}{2}\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x} = \boxed{\dfrac{5}{2}}$

(c.) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{\sin 3x} = \lim\limits_{x\to 0} \dfrac{4\sin 4x}{4\cancel{x}} \cdot \dfrac{3\cancel{x}}{3\sin 3x} = \lim\limits_{x\to 0} \dfrac{4\sin 4x}{4x} \cdot \lim\limits_{x\to 0} \dfrac{3x}{3\sin 3x}$

$$= \frac{4}{3} \lim_{x\to 0} \frac{\sin 4x}{4x} \lim_{x\to 0} \frac{3x}{\sin 3x} = \boxed{\frac{4}{3}}$$

(d.) $\lim_{x\to 0} \frac{\tan 3x}{\sin 2x} = \lim_{x\to 0} \frac{3\tan 3x}{3\cancel{x}} \cdot \frac{2\cancel{x}}{2\sin 2x} = \lim_{x\to 0} \frac{3\tan 3x}{3x} \cdot \lim_{x\to 0} \frac{2x}{2\sin 2x}$

$= \frac{3}{2} \lim_{x\to 0} \frac{\tan 3x}{3x} \cdot \lim_{x\to 0} \frac{2x}{\sin 2x} = \boxed{\frac{3}{2}}$

$6 = (3)(2)$ $\sin 2x = 2\sin x \cos x$

(e.) $\lim_{x\to 0} \frac{\sin 6x}{\sin 3x}$

$= \lim_{x\to 0} \frac{2\cancel{\sin(3x)}\cos(3x)}{\cancel{\sin(3x)}} = \lim_{x\to 0} 2\cos(3x) = \boxed{2}$

$\frac{d}{dx}$ - derivative with respect to x

Derivatives

$$\boxed{\frac{d}{dx}\sin x = \cos x}$$

$$\boxed{\frac{d}{dx}\cos x = -\sin x}$$

$$\boxed{\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} = \sec^2 x}$$

$$\boxed{\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x} = -\csc^2 x}$$

$$\boxed{\frac{d}{dx}\csc x = -\csc x \cot x}$$

$$\boxed{\frac{d}{dx}\sec x = \sec x \tan x}$$

$\tan x = \frac{\sin x}{\cos x}$

$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$

$= \frac{(\cos x)(\cos x) - \sin x(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

$\csc x = \frac{1}{\sin x}$

$(\csc x)' = \left(\frac{1}{\sin x}\right)' = \frac{(1)'\sin x - 1(\sin x)'}{\sin^2 x}$

$= \frac{-\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$

$= -\cot x \cdot \csc x$

Example 2. Find $\frac{dy}{dx}$ (find the derivative of y).

(a.) $y = \cos x - 2\tan x$

$y' = (\cos x - 2\tan x)' = (\cos x)' - 2(\tan x)'$

$= \boxed{-\sin x - 2\sec^2 x}$

(b.) $y = 2x(\sqrt{x} - \cot x) = 2x\underbrace{\sqrt{x}}_{x^{3/2}} - 2x\cot x = 2x^{3/2} - 2x\cot x$

$y' = (2x^{3/2} - 2x\cot x)' = 2(x^{3/2})' - 2(x\cot x)'$

$= 2\left(\frac{3}{2}\right)x^{3/2-1} - 2\left((x)'\cot x + x(\cot x)'\right)$

$= \boxed{3x^{1/2} - 2(\cot x - x\csc^2 x)}$

(c.) $y = x\csc x = \frac{x}{\sin x}$

$y' = \left(\frac{x}{\sin x}\right)' = \frac{(x)'\sin x - x(\sin x)'}{\sin^2 x} = \boxed{\frac{\sin x - x\cos x}{\sin^2 x}}$

$y' = (x\csc x)' = (x)'\csc x + x(\csc x)'$

$= \boxed{\csc x - x\csc x\cot x}$

Example 3. Find the points on the curve $y = \frac{\cos x}{2 + \sin x}$ at which the tangent is horizontal. $(0 \le x \le 2\pi)$

horizontal tangent $\rightarrow$ angle is zero $\rightarrow$ slope is zero

$$\text{slope} = y' = \left(\frac{\cos x}{2+\sin x}\right)' = \frac{(\cos x)'(2+\sin x) - (\cos x)(2+\sin x)'}{(2+\sin x)^2} = 0$$

$2 + \sin x$ is never zero

$$(-\sin x)(2+\sin x) - \cos x(\cos x) = 0$$

$$-2\sin x \underbrace{- \sin^2 x - \cos^2 x}_{-1} = 0$$

$$-2\sin x - 1 = 0 \Rightarrow \sin x = -\frac{1}{2}$$

$$x_1 = \pi + \frac{\pi}{6}, \quad x_2 = 2\pi - \frac{\pi}{6}$$

$$x_1 = \frac{7\pi}{6}, \quad x_2 = \frac{11\pi}{6}$$

$$y\left(\frac{7\pi}{6}\right) = \frac{\cos\frac{7\pi}{6}}{2+\sin\frac{7\pi}{6}} = \frac{-\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = \frac{-\frac{\sqrt{3}}{2}}{\frac{3}{2}} = -\frac{1}{\sqrt{3}}$$

$$y\left(\frac{11\pi}{6}\right) = \frac{\cos\frac{11\pi}{6}}{2+\sin\frac{11\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{3/2} = \frac{1}{\sqrt{3}}$$

$\boxed{\left(\frac{7\pi}{6}, -\frac{1}{\sqrt{3}}\right)}$ and $\boxed{\left(\frac{11\pi}{6}, \frac{1}{\sqrt{3}}\right)}$