1. Quiz 4 over 3.2 and 3.4
2. HW over 3.2, 3.4, 3.5, 3.6, 3.7 due Wednesday, Oct. 12, 11:55PM
3. Tests will be returned [redacted] during recitations
4. Retest on Friday, Oct. 7 7-9PM in BLOC 102

Section 3.5 The Chain Rule

If the derivatives $g'(x)$ and $f'(g(x))$ both exist, and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F'(x)$ exists an is given by the product

$$\boxed{[f(g(x))]' = F'(x) = f'(g(x))g'(x)}$$

f is the outer function
g is the inner function

Example 1. Suppose that $F(x) = f(g(x))$, where $g(2) = 5$, $g'(2) = 4$, $f(2) = 3$, $f'(2) = -2$, and $f'(5) = 11$. Find $F'(2)$.

$$F'(x) = f'(g(x))g'(x)$$

$$F'(2) = f'(\underbrace{g(2)}_{5})\,\underbrace{g'(2)}_{4}$$

$$= \underbrace{f'(5)}_{11}(4)$$

$$= (11)(4) = \boxed{44}$$

If n is any real number and $u = g(x)$ is differentiable, then

$$\boxed{\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)}$$

General Power Rule

$f(x) = x^2+3x-2, \quad g(x) = \sin x$

$f(g(x)) = \sin^2 x + 3\sin x - 2$ (plug $\sin x$ for x into the equation of f)

$g(f(x)) = \sin(x^2+3x-2)$

$f(x) = \tan x, \quad g(x) = \sqrt{x}$

$f(g(x)) = \tan(\sqrt{x})$

$g(f(x)) = \sqrt{\tan x}$

Example 2. Find the derivative of each function

$(\sec x)' = \sec x \tan x$

1. $y = \underbrace{\sec}_{\text{outer}}\overbrace{(2x)}^{\text{inner}}$

$$y' = \sec(2x)\tan(2x)\,(2x)'$$
$$= \boxed{2\sec(2x)\tan(2x)}$$

2. $y = \overbrace{\sin}^{\text{outer}}\underbrace{(x^2)}_{\text{inner}}$

$$y' = \cos(x^2)\,(x^2)'$$
$$= \boxed{2x\cos(x^2)}$$

3. $y = (\underbrace{1+\cos^2 x}_{\text{inner}})^{6}$ outer

Do the General Power Rule first.

$$y' = 6(1+\cos^2 x)^5 (1+\underbrace{\cos}_{\text{inner}}{}^{2} x)' \quad \text{outer}$$

$$= 6(1+\cos^2 x)^5 (2\cos x)(\cos x)'$$

$$= \boxed{6(1+\cos^2 x)^5 (2\cos x)(-\sin x)}$$

4. $y = (1+\sqrt{x^2+2})^3$

General Power Rule first:

$$y' = 3(1+\sqrt{x^2+2})^2 (1+\sqrt{x^2+2})'$$

$$= 3(1+\sqrt{x^2+2})^2 ((x^2+2)^{1/2})'$$

General Power Rule again

$$= 3(1+\sqrt{x^2+2})^2 \frac{1}{2}(x^2+2)^{\frac{1}{2}-1} (x^2+2)'$$

$$= \frac{3}{\cancel{2}}(1+\sqrt{x^2+2})^2 (x^2+2)^{-1/2} (\cancel{2}x)$$

$$= \boxed{3(1+\sqrt{x^2+2})^2 (x^2+2)^{-1/2}(x)}$$

5. $y = \sqrt[4]{\frac{t^3+1}{t^3-1}} = \underline{\left(\frac{t^3+1}{t^3-1}\right)^{1/4}} = \frac{(t^3+1)^{1/4}}{(t^3-1)^{1/4}} = (t^3+1)^{1/4}(t^3-1)^{-1/4}$

General Power Rule first:

$$y' = \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{1/4-1} \left(\frac{t^3+1}{t^3-1}\right)'$$

$$= \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4} \frac{(t^3+1)'(t^3-1) - (t^3-1)'(t^3+1)}{(t^3-1)^2}$$

$$= \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4} \frac{3t^2(t^3-1) - 3t^2(t^3+1)}{(t^3-1)^2}$$

$$= \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4} \frac{\cancel{3t^2t^3} - 3t^2 - \cancel{3t^2t^3} - 3t^2}{(t^3-1)^2}$$

$$= \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4} \frac{-6t^2}{(t^3-1)^2} = -\frac{3t^2}{2} \frac{(t^3+1)^{-3/4}}{(t^3-1)^{2-3/4}}$$

$$= -\frac{3t^2}{2} \frac{(t^3+1)^{-3/4}}{(t^3-1)^{5/4}} = \boxed{-\frac{3t^2}{2(t^3+1)^{3/4}(t^3-1)^{5/4}}}$$

6. $y = \sin^2(\cos 4x)$ General Power Rule first:

$$y' = 2\sin(\cos 4x)\,[\underbrace{\sin}_{\text{outer}}(\overbrace{\cos 4x}^{\text{inner}})]'$$

$$= 2\sin(\cos 4x)\cos(\cos 4x)\,[\underbrace{\cos}_{\text{outer}}\overbrace{4x}^{\text{inner}}]'$$

$$= 2\sin(\cos 4x)\cos(\cos 4x)(-\sin 4x)\underbrace{(4x)'}_{4}$$

$$= \boxed{-8\sin 4x\,[\sin(\cos 4x)][\cos(\cos 4x)]}$$

Example 3. Find the equation of the tangent line to the curve $y = \dfrac{8}{\sqrt{4+3x}}$ at the point (4,2).

$$y = \overbrace{f'(a)}^{-3/16}(x-\overbrace{a}^{4}) + \overbrace{f(a)}^{2}$$ tangent line to $y=f(x)$ @ $(a, f(a))$

$f(x) = 8(4+3x)^{-1/2}$ $a=4$, $f(a)=2$

$$f'(x) = 8\left(-\tfrac{1}{2}\right)(4+3x)^{-1/2-1}(4+3x)'$$

$$= -4(4+3x)^{-3/2}(3) = \frac{-12}{(4+3x)^{3/2}}$$

$$f'(4) = \frac{-12}{(4+4(3))^{3/2}} = -\frac{12}{16^{3/2}} = -\frac{12}{64} = -\frac{3}{16}$$

$$\boxed{y = -\frac{3}{16}(x-4)+2}$$