Section 3.6 **Implicit differentiation.**

Some functions are defined implicitly by a relation between x and y, where x is the independent variable and y depends on x. In order to find the derivative of y with respect to x, we can use the method of **implicit differentiation**. This consists of differentiating both sides of the relation with respect to x and then solving the resulting equation for y'.

Example 1. Find dy/dx by implicit differentiation.

1. $(x^2 - xy + y^3)' = (\overset{0}{8})'$ $\quad y = y(x)$

$(\overset{2x}{x^2})' - (xy)' + (y^3)' = 0$

$2x - (x'y + y'x) + 3y^2y' = 0$ Keep y' as is.

$2x - (y + xy') + 3y^2y' = 0$ solve for y'!

$2x - y - xy' + 3y^2y' = 0$

$$\frac{y'(3y^2 - x)}{3y^2-x} = \frac{y - 2x}{3y^2-x}$$

$$\boxed{y' = \frac{y-2x}{3y^2-x}}$$

2. $\left(\dfrac{y}{x-y}\right)' = (x^2+1)'$

$$(x-y)^2\,\frac{y'(x-y) - (x-y)'y}{(x-y)^2} = 2x\,(x-y)^2$$

$y'(x-y) - (\overset{1}{x'} - y')y = 2x(x-y)^2$

$y'(x-y) - (1-y')y = 2x(x-y)^2$

$xy' - \cancel{yy'} - y + \cancel{yy'} = 2x(x-y)^2$

$$\frac{xy'}{x} = \frac{2x(x-y)^2 + y}{x}$$

$$\boxed{y' = \frac{2x(x-y)^2+y}{x} = 2(x-y)^2 + \frac{y}{x}}$$

3. $\sqrt{x+y}+\sqrt{xy}=6$

$$\left((x+y)^{1/2}+(xy)^{1/2}\right)'=(6)'$$

$$\frac{1}{2}(x+y)^{1/2-1}(x+y)'+\frac{1}{2}(xy)^{1/2-1}(xy)'=0$$

$$\frac{1}{2}(x+y)^{-1/2}(1+y')+\frac{1}{2}(xy)^{-1/2}(x'y+xy')=0$$

$$\frac{1}{2}(x+y)^{-1/2}(1+y')+\frac{1}{2}(xy)^{-1/2}(y+xy')=0$$

$$\frac{1}{2}(x+y)^{-1/2}y'+\frac{1}{2}(xy)^{-1/2}xy'=-\frac{1}{2}(x+y)^{-1/2}-\frac{1}{2}(xy)^{-1/2}y$$

$$y'\left(\frac{1}{\cancel{2}}(x+y)^{-1/2}+\frac{x}{\cancel{2}}(xy)^{-1/2}\right)=-\frac{1}{\cancel{2}}(x+y)^{-1/2}-\frac{y}{\cancel{2}}(xy)^{-1/2}$$

$$\boxed{y'=\frac{-(x+y)^{-1/2}-y(xy)^{-1/2}}{(x+y)^{-1/2}+x(xy)^{-1/2}}}$$

4. $(x\sin y+\overset{\text{outer}}{\cos}\,\underset{\text{inner}}{2y})'=(\overset{\text{outer}}{\cos}\,\underset{\text{inner}}{y})'$

$$x'\sin y+x(\underset{\text{outer}}{\sin}\,\underset{\text{inner}}{y})'+(-\sin(2y))(2y)'=-\sin y(y')$$

$$\sin y+x(\cos y)(y')-\sin(2y)(2y')=-\sin y(y')$$

$$\sin y=2y'\sin 2y-xy'\cos y-y'\sin y$$

$$\sin y=y'(2\sin 2y-x\cos y-\sin y)$$

$$\boxed{y'=\frac{\sin y}{2\sin 2y-x\cos y-\sin y}}$$

$x = x(y)$

Example 2. Let y be the independent variable and x be the dependent variable. Use implicit differentiation to find dx/dy if

$$\frac{d}{dy}\left[(x^2+y^2)^2\right] = \frac{d}{dy}\left[4x^2y\right]$$

Differentiate for y.
$\frac{d}{dy}$ – take the derivative with respect to y

$$2(x^2+y^2)\frac{d}{dy}(x^2+y^2) = 4\left(\frac{d}{dy}x^2(y) + x^2\frac{d}{dy}y\right)$$

$$\cancel{2}(x^2+y^2)(\cancel{2}x\,x' + \cancel{2}y) = \cancel{4}(y(2x)x' + x^2)$$

$$(x^2+y^2)\,x x' - 2xy x' = x^2 - (x^2+y^2)y$$

$$x'\left(x(x^2+y^2) - 2xy\right) = x^2 - y(x^2+y^2)$$

$$\boxed{x' = \frac{x^2 - y(x^2+y^2)}{x(x^2+y^2) - 2xy}}$$

Definition. Two curves are called orthogonal if at each point of intersection their tangent lines are perpendicular.

lines $y = k_1x + b_1$, $y = k_2x + b_2$ $\perp$ if and only if $k_1k_2 = -1$

Example 3. Show that the curves $\underbrace{x^2 - y^2 = 5}_{y'}$ and $\underbrace{4x^2 + 9y^2 = 72}_{y'}$ are orthogonal.

$$(x^2-y^2)' = (5)'$$
$$2x - 2yy' = 0$$
$$y' = \frac{x}{y}$$
$$k_1 = \frac{x}{y}$$

$$(4x^2+9y^2)' = (72)'$$
$$8x + 18yy' = 0$$
$$y' = -\frac{8x}{18y} = -\frac{4x}{9y}$$
$$k_2 = -\frac{4x}{9y} \Rightarrow k_1k_2 = -\frac{4x^2}{9y^2}$$

Points of intersection:

$$\begin{cases} x^2 - y^2 = 5 \longrightarrow & x^2 = 5 + y^2 \\ 4x^2 + 9y^2 = 72 \end{cases}$$

$$4(5+y^2) + 9y^2 = 72$$
$$20 + 4y^2 + 9y^2 = 72$$
$$13y^2 = 52 \rightarrow y^2 = 4$$
$$y = \pm 2$$
$$x^2 = 5 + 4 = 9$$
$$x = \pm 3$$

$(-3,-2), (-3,2), (3,-2), (3,2)$

$(-3,-2)$: $k = -\frac{4(-3)^2}{9(-2)^2} = -1$

$(-3,2)$: $k = -\frac{4(-3)^2}{9(2)^2} = -1$

$(3,-2)$: $k = \frac{-4(3)^2}{9(-2)^2} = -1$

$(3,2)$: $k = \frac{-4(3)^2}{9(2)^2} = -1$

Orthogonal!