Section 3.8 **Higher derivatives**

$$\frac{d^2 f}{dx^2} = f''(x) = [f'(x)]'$$ the 2nd order derivative

$$\frac{d^3 f}{dx^3} = f'''(x) = [f''(x)]'$$ the 3rd derivative

$$\boxed{\frac{d^n f}{dx^n} = f^{(n)}(x) = [f^{(n-1)}(x)]'}$$ the n-th derivative

Example 1. Find $f''(x)$ for the function $f(x) = \tan^3(2x-1)$.

$$f'(x) = 3\tan^2(2x-1)[\tan(2x-1)]'$$
$$= 3\tan^2(2x-1)\sec^2(2x-1)(2)$$
$$= 6\tan^2(2x-1)\sec^2(2x-1)$$

$$f'' = 6(\tan^2(2x-1)\sec^2(2x-1))'$$
$$= 6((\tan^2(2x-1))'\sec^2(2x-1) + \tan^2(2x-1)(\sec^2(2x-1))')$$
$$= 6[2\tan(2x-1)(\tan(2x-1))'\sec^2(2x-1) + \tan^2(2x-1)\,2\sec(2x-1)(\sec(2x-1))']$$
$$= 12[\tan(2x-1)\,2\sec^2(2x-1)\sec^2(2x-1) + \tan^2(2x-1)\sec(2x-1)\sec(2x-1)\tan(2x-1)(2)]$$
$$= \boxed{24\tan(2x-1)\sec^2(2x-1)[\sec^2(2x-1) + \tan^2(2x-1)]}$$

Example 2. Find $\frac{d^3}{dx^3}\left(\frac{1-x}{1+x}\right)$

$\frac{1-x}{1+x}$ improper fraction (power of x on the top ≥ power of x on the bottom)

separate the whole part of the fraction.

$\frac{1-x}{x+1}$ need $1+x$ on the top

$$= \frac{[(1-x)+(x+1)]-(x+1)}{x+1} = \frac{-(x+1)}{x+1} + \frac{1-x+(x+1)}{x+1} = -1 + \frac{1-x+x+1}{x+1}$$
$$= -1 + \frac{2}{x+1} = -1 + 2(x+1)^{-1}$$

$$f(x) = -1 + 2(x+1)^{-1}$$
$$f'(x) = -2(x+1)^{-1-1} = -2(x+1)^{-2}$$
$$f''(x) = -2(-2)(x+1)^{-2-1} = 4(x+1)^{-3}$$
$$f'''(x) = \boxed{-12(x+1)^{-4}}$$

Example 3. Find a formula for $f^{(n)}(x)$ for the following functions:

(a.) $f(x) = x^{\boxed{4}} - 3x^3 + 16x$

$$f'(x) = 4x^3 - 9x^2 + 16$$
$$f''(x) = 12x^2 - 18x$$
$$f'''(x) = 24x - 18$$
$$f^{\boxed{\text{IV}}}(x) = 24 = 4(3)(2)(1)$$
$$f^{\text{V}}(x) = 0$$

(b.) $f(x) = \sqrt{x} = x^{1/2}$

$$f'(x) = \frac{1}{2} x^{-1/2}$$
$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-1/2-1} = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-3/2}$$
$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-5/2}$$

$n = 2$ $\quad (1)(1)$, with $\overbrace{(1)}^{2(2)-3}$

$n = 3$ $\quad (1)(3)$, with $\underbrace{(3)}_{2(3)-3}$

$n = 4$ $\quad (1)(3)(5)$, with $\underbrace{(5)}_{4(2)-3}$

$$\boxed{f^{(n)}(x) = \frac{1}{2}(-1)^{n+1} \frac{1(3)(5)\cdot\ldots\cdot(2n-3)}{2^{n-1}} x^{-(2n-1)/2}}$$

$n = 4$

$$f^{\text{IV}}(x) = -\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2(4)-3}{2}\right) x^{-(2(4)-1)/2}$$
$$= -\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) x^{-7/2}$$

(c.) $f(x) = x^{\boxed{n}}$ - polynomial of degree n

$$f^{(n)}(x) = n(n-1)(n-2)\cdot\ldots\cdot(2)(1) = n!$$

(c.) $f(x) = \dfrac{1}{(1-x)^2} = (1-x)^{-2}$

$f'(x) = -2(1-x)^{-2-1}(1-x)'$

$= -2(1-x)^{-3}(-1)$

$f'(x) = 2(1-x)^{-3}$

$f''(x) = 2(-3)(1-x)^{-4}(-1)$

$= 2(3)(1-x)^{-4}$

$f'''(x) = 2(3)(-4)(1-x)^{-5}(-1)$

$= 2(3)(4)(1-x)^{-5}$

$f^{(n)}(x) = 2(3)(4)\ldots(n+1)(1-x)^{-(n+2)}$

$\boxed{f^{(n)}(x) = (n+1)!\,(1-x)^{-(n+2)}}$

$\boxed{n! = (1)(2)\ldots(n-1)(n)}$

Example 4. Find $f^{(25)}(x)$ if $f(x) = x\sin x$.

$f'(x) = \sin x + x\cos x$

$f''(x) = \cos x + \cos x + x(-\sin x)$

$= 2\cos x - x\sin x$

$f'''(x) = -2\sin x - (\sin x + x\cos x)$

$= -3\sin x - x\cos x$

$f^{IV}(x) = -3\cos x - (\cos x - x\sin x)$

$= -4\cos x + x\sin x$

$f^{V}(x) = 4\sin x + \sin x + x\cos x$

$= 5\sin x + x\cos x$

$5 = 4\cdot 1 + 1$

$25 = (6)(4) + 1$

$f^{(25)}(x) = 25\sin x + x\cos x$

$f^{(35)}(x) = -35\sin x - x\cos x$

$35 = 8(4) + 3$

10/12: • Quiz over 3.4-3.7
• due date for the homework over 3.2, 3.4-3.7

Acceleration.

Let $s = s(t)$ be the position function of an object that moves in a straight line.

The instantaneous rate of change of velocity with respect to time is called acceleration $a(t)$ of the object. Thus, the acceleration function is the derivative of the velocity function. Therefore

$$\boxed{a(t) = v'(t) = s''(t).}$$

Example 5. The equation of motion of a particle is $s(t) = 2t^3 - 7t^2 + 4t + 1$, where s is measured in meters and t in seconds. Find the acceleration as a function of time. What is acceleration after 1 s?

$s(t) = 2t^3 - 7t^2 + 4t + 1$

velocity: $V(t) = s'(t) = 6t^2 - 14t + 4$

acceleration: $\boxed{a(t) = V'(t) = 12t - 14}$

$a(1) = 12 - 14 = \boxed{-2 \ m/sec^2}$

For the vector function $\vec{r}(t) =< x(t), y(t) >$

$$\vec{r}\,'(t) =< x'(t), y'(t) >$$

$$\vec{r}\,''(t) = [\vec{r}\,'(t)]' =< x''(t), y''(t) >$$

If $\vec{r}(t) =< x(t), y(t) >$ represents the position of an object then the acceleration vector is

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) =< x''(t), y''(t) >$$

Example 5. Find the acceleration at $t = 2$ if $\vec{r}(t) = \sqrt{t^2 + 5}\vec{\imath} + t\vec{\jmath}$.

$\vec{r}(t) = <\sqrt{t^2+5}, t> \; = \; <(t^2+5)^{1/2}, t>$

$\vec{r}\,'(t) = <\frac{1}{2}(t^2+5)^{-1/2}(t^2+5)', 1>$

$= <\frac{1}{\cancel{2}}(t^2+5)^{-1/2}(\cancel{2}t), 1>$

$= <t(t^2+5)^{-1/2}, 1>$

acceleration vector $\vec{a}(t) = \vec{r}\,''(t) = <(t)'(t^2+5)^{-1/2} + t\underbrace{[(t^2+5)^{-1/2}]'}, 0>$

$= <(t^2+5)^{-1/2} + t\left(-\frac{1}{2}\right)(t^2+5)^{-3/2}(t^2+5)', 0>$

$= <(t^2+5)^{-1/2} - \frac{t}{\cancel{2}}(t^2+5)^{-3/2}(\cancel{2}t), 0>$

$= <(t^2+5)^{-1/2} - t^2(t^2+5)^{-3/2}, 0>$

$\vec{a}(2) = <(2^2+5)^{-1/2} - 4(2^2+5)^{-3/2}, 0>$

$= <\frac{1}{3} - \frac{4}{9\sqrt{9}}, 0>$

$= <\frac{1}{3} - \frac{4}{27}, 0> = <\frac{9-4}{27}, 0> = <\frac{5}{27}, 0>$

acceleration $a = |\vec{a}(2)| = \boxed{\frac{5}{27}}$

Implicit second derivative.

Example 6. Find $\frac{d^2y}{dx^2}$ if $x^2+6xy+y^2=8$.

product rule

$$\frac{d}{dx}(x^2+6xy+y^2)=\frac{d}{dx}(8)$$

$$\underbrace{2x+6\left[\frac{d(x)}{dx}y+x\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0}_{2}$$

$$x+3\left(y+x\frac{dy}{dx}\right)+y\frac{dy}{dx}=0$$

$$x+3y+3x\frac{dy}{dx}+y\frac{dy}{dx}=0 \qquad \text{solve for } \frac{dy}{dx}$$

$$(3x+y)\frac{dy}{dx}=-3y-x$$

$$\frac{dy}{dx}=-\frac{x+3y}{3x+y}$$

$$\frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{x+3y}{3x+y}\right)=-\frac{\frac{d}{dx}(x+3y)\,(3x+y)-(x+3y)\frac{d(3x+y)}{dx}}{(3x+y)^2}$$

$$=-\frac{\left(1+3\frac{dy}{dx}\right)(3x+y)-(x+3y)\left(3+\frac{dy}{dx}\right)}{(3x+y)^2}$$

$$=-\frac{\cancel{3x}+y+\underbrace{9x\frac{dy}{dx}}+\cancel{3y\frac{dy}{dx}}-\cancel{3x}-\underbrace{x\frac{dy}{dx}}-9y-\cancel{3y\frac{dy}{dx}}}{(3x+y)^2}$$

$$=-\frac{8x\frac{dy}{dx}-8y}{(3x+y)^2}=+8\cdot\frac{+x\frac{x+3y}{3x+y}+y}{(3x+y)^2}=8\,\frac{\frac{x(x+3y)+y(3x+y)}{3x+y}}{(3x+y)^2}$$

$$=8\,\frac{x^2+3xy+3xy+y^2}{(3x+y)^3}=8\,\frac{\overbrace{x^2+6xy+y^2}^{8}}{(3x+y)^3}$$

very original equation: $x^2+6xy+y^2=8$

$$=8\,\frac{8}{(3x+y)^3}=\boxed{\frac{64}{(3x+y)^3}}$$