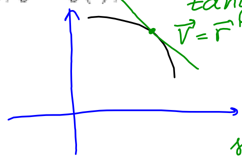
Section 3.9 **Slopes and tangents to parametric curves**

Suppose that the curve C is given by parametric equations $x = x(t)$, $y = y(t)$, then

slope of the tangent line $\boxed{\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{y'(t)}{x'(t)}}$



tangent vector $\vec{v} = \vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$

slope of the tangent line

slope $= \dfrac{y'(t)}{x'(t)}$

Example 1. Find an equation of the tangent to the curve $x(t) = t \sin t$, $y(t) = t \cos t$ at the point corresponding to $t = \pi$.

Equation $\quad y = \dfrac{dy}{dx}(\pi)\,(x - x(\pi)) + y(\pi)$

$x(\pi) = \pi \underbrace{\sin \pi}_{0} = 0, \quad y(\pi) = \pi \underbrace{\cos \pi}_{-1} = -\pi$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{[t \cos t]'}{[t \sin t]'} = \frac{\cos t - t \sin t}{\sin t + t \cos t}$$

$$\frac{dy}{dx}(\pi) = \frac{\cos \pi - \pi \sin \pi}{\sin \pi + \pi \cos \pi} = \frac{-1}{-\pi} = \frac{1}{\pi}$$

$$y = \frac{1}{\pi}(x - 0) - \pi$$

$$\boxed{y = \frac{x}{\pi} - \pi}$$

Example 2. Find the points on the curve $x = t(t^3 - 3)$, $y = 3(t^3 - 3)$, where the tangent is vertical or horizontal.

$x(t) = t^4 - 3t$

$y(t) = 3t^3 - 9$

slope of a tangent line $\quad \dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{9t^2}{4t^3 - 3}$

horizontal tangent $\rightarrow$ slope is zero $\rightarrow \quad y'(t) = 0$

$9t^2 = 0, \quad t = 0$

Point corresponding to $t = 0$: $\quad x(0) = 0, \; y(0) = -9$

$\boxed{\text{horizontal tangent @ } (0, -9)}$

vertical tangent is when $\quad x'(t) = 0$ or $4t^3 - 3 = 0$

$t = \sqrt[3]{\frac{3}{4}}$

Point corresponding to $t = \sqrt[3]{\frac{3}{4}}$:

$$x\left(\sqrt[3]{\tfrac{3}{4}}\right) = \sqrt[3]{\tfrac{3}{4}}\left(\left(\sqrt[3]{\tfrac{3}{4}}\right)^3 - 3\right) = \sqrt[3]{\tfrac{3}{4}}\left(\tfrac{3}{4} - 3\right) = \sqrt[3]{\tfrac{3}{4}}\;\frac{3 - 12}{4} = -\frac{9}{4}\left(\sqrt[3]{\tfrac{3}{4}}\right)$$

$$y\left(\sqrt[3]{\tfrac{3}{4}}\right) = 3\left(\left(\sqrt[3]{\tfrac{3}{4}}\right)^3 - 3\right) = 3\left(\tfrac{3}{4} - 3\right) = -\frac{27}{4}$$

$$\boxed{\text{vertical tangent @ } \left(-\tfrac{9}{4}\sqrt[3]{\tfrac{3}{4}}, \; -\tfrac{27}{4}\right)}$$

Example 3. At what points on the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent parallel to the line with the equations $\underbrace{x = -7t,\ y = 12t - 5}_{\text{slope} = -\frac{12}{7}}$?

Find the points on the curve where the slope of the tangent is $-\frac{12}{7}$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(6t^2)'}{(t^3+4t)'} = \frac{\cancel{12}t}{3t^2+4} = -\frac{\cancel{12}}{7}$$

$$7t = -(3t^2+4)$$

$$3t^2+7t+4=0$$

$$t_1 = \frac{-7+\sqrt{49-4(4)(3)}}{6} = \frac{-7+1}{6} = -1$$

$$t_2 = \frac{-7-1}{6} = -\frac{8}{6} = -\frac{4}{3}$$

$t=-1$:

$x(-1) = (-1)^3+4(-1) = -5$

$y(-1) = 6(-1)^2 = 6$

$\boxed{(-5,6)}$

$t = -\frac{4}{3}$

$$x\left(-\tfrac{4}{3}\right) = \left(-\tfrac{4}{3}\right)^3 + 4\left(-\tfrac{4}{3}\right) = \underbrace{-\frac{64}{27} - \frac{16}{3}}^{-\frac{64+9(16)}{27}} = -\frac{208}{27}$$

$$y\left(-\tfrac{4}{3}\right) = 6\left(-\tfrac{4}{3}\right)^2 = \frac{6(16)}{9} = \frac{2(16)}{3} = \frac{32}{3}$$

$\boxed{\left(-\frac{208}{27}, \frac{32}{3}\right)}$