Section 4.2 **Inverse functions**

Definition. A function f with domain A is called **one-to-one function** if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Horizontal line test. A function is one-to-one if and only if no horizontal line intersects its graph more that once.

Example 1. Determine which of the following functions is one-to-one:

1. $f(x) = x + 5$

2. $g(x) = x^2 - 2x + 5$

3. $h(x) = x^3 - 1$

4. $p(x) = x^4 + 5$

Definition. Let f be one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Longleftrightarrow f(x) = y$$

for any y in B.

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

Let f be one-to-one function with domain A and range B. If $f(a) = b$, then $f^{-1}(b) = a$.

Cancellation equations

$$f^{-1}(f(x)) = x \text{ for every } x \in A$$
$$f(f^{-1}(x)) = x \text{ for every } x \in B$$

How to find the inverse function of a one-to-one function f

1. Write $y = f(x)$
2. Solve this equation for x in terms of y.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

The graph of f^{-1} is obtained by the reflecting the graph f about the line $y = x$.

Example 2. Show that the function $f(x) = \dfrac{1+3x}{5-2x}$ is one-to-one and find $f^{-1}(y)$.

Theorem. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

Theorem. If f is a one-to-one differentiable function with inverse function $g = f^{-1}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and

$$g'(a) = \frac{1}{f'(g(a))}$$

Example 3. Find $g'(4)$, where g is the inverse function of the function $f(x) = 3 + x + \mathrm{e}^x$.