Quiz 6 over 3.8-3.10
HW over 3.8-3.10 is due 10/19, 11:55 PM
HW over 3.11, 4.1 is due 10/26, 11:55 PM

Chapter 4. **Inverse functions: exponential, logarithmic, and inverse trigonometric functions**

Section 4.1 **Exponential functions and their derivatives**

An exponential function is a function of the form

$$\boxed{f(x) = a^x, \quad a>0, \quad a \neq 1}$$

where a is a positive constant. It is defined in five stages:

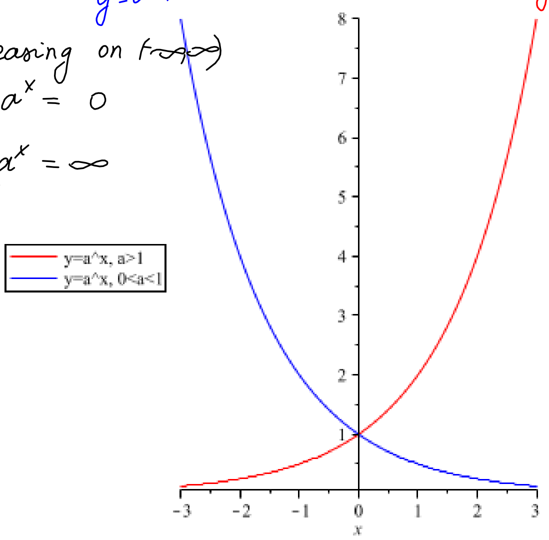
- If $x = n$, a positive integer, then
$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$$
- $a^0 = 1$
- If $x = -n$, n is a positive integer, then
$$a^{-n} = \frac{1}{a^n}$$
- If $x = \frac{p}{q}$ is a rational number, where p and q are integers and $q > 0$, then
$$a^{p/q} = \sqrt[q]{a^p}$$
- If x is an irrational number, then
$$a^x = \lim_{r \to x} a^r \qquad r \text{ rational}$$

$y=a^x$, $0<a<1$

- decreasing on $(-\infty,\infty)$
- $\lim_{x\to\infty} a^x = 0$
- $\lim_{x\to-\infty} a^x = \infty$

$y=a^x$, $a>1$

- increasing on $(-\infty,\infty)$
- $\lim_{x\to\infty} a^x = \infty$
- $\lim_{x\to-\infty} a^x = 0$



- Domain $(-\infty,\infty)$
- Range $(0,\infty)$
- Continuous on $(-\infty,\infty)$

Theorem. If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is continuous function with domain $(-\infty, \infty)$ and range $(0, \infty)$.

- If $0 < a < 1$, $f(x) = a^x$ is decreasing function
- if $a > 1$, $f(x) = a^x$ is increasing function
- If $a, b > 0$ and x, y are reals, then

$$\boxed{1.\ a^{x+y} = a^x a^y \quad 2.\ a^{x-y} = \frac{a^x}{a^y} \quad 3.\ (a^x)^y = a^{xy} \quad 4.\ (ab)^x = a^x b^x}$$

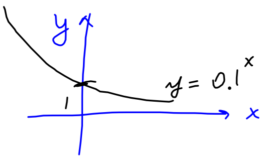
- If $0 < a < 1$, $\boxed{\lim_{x\to-\infty} a^x = \infty, \ \lim_{x\to\infty} a^x = 0}$
- If $a > 1$, $\boxed{\lim_{x\to-\infty} a^x = 0, \ \lim_{x\to\infty} a^x = \infty}$

Example 1. Find each limit.

1. $\lim_{x\to\infty} (0.1)^x = \boxed{0}$

   $0.1 < 1$

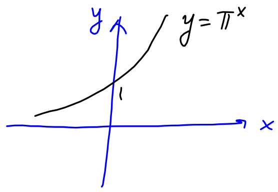
   $\lim_{x\to-\infty} (.1)^x = \infty$



2. $\lim_{x\to-\infty} \pi^x = \boxed{0}$

   $\pi = 3.14 > 1$

   $\lim_{x\to\infty} \pi^x = \infty$



3. $\lim_{x\to-\infty} \dfrac{2^{3x} - 2^{-3x}}{2^{3x} + 2^{-3x}}$

$$\lim_{x\to-\infty} 2^{3x} = \lim_{x\to-\infty} (2^3)^x = \lim_{x\to-\infty} 8^x = 0$$

$$\lim_{x\to-\infty} 2^{-3x} = \lim_{x\to-\infty} (2^{-3})^x = \lim_{x\to-\infty} \left(\frac{1}{8}\right)^x = \infty \quad (\text{get rid of it}).$$

$$= \lim_{x\to-\infty} \frac{2^{3x} - \frac{1}{2^{3x}}}{2^{3x} + \frac{1}{2^{3x}}} = \lim_{x\to-\infty} \frac{\frac{2^{3x}\cdot 2^{3x} - 1}{2^{3x}}}{\frac{2^{3x}\cdot 2^{3x}+1}{2^{3x}}} = \lim_{x\to-\infty} \frac{2^{3x+3x} - 1}{\cancel{2^{3x}}} \cdot \frac{\cancel{2^{3x}}}{2^{3x+3x}+1}$$

$$= \lim_{x\to-\infty} \frac{2^{6x}-1}{2^{6x}+1} = \frac{\left(\cancel{\lim_{x\to-\infty} 2^{6x}}\, 0\right) - 1}{\left(\cancel{\lim_{x\to-\infty} 2^{6x}}\, 0\right) + 1} = \boxed{-1}$$

Derivative of exponential function. If $f(x) = a^x$, then

$$f'(x) = (a^x)' = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = \lim_{h\to 0}\frac{a^x(a^h-1)}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h} = a^x f'(0)$$

$$\boxed{(a^x)' = a^x f'(0)}$$ where $f(x)=a^x$

Let e be a number such that

$$\boxed{\lim_{h\to 0}\frac{e^h-1}{h} = 1}$$

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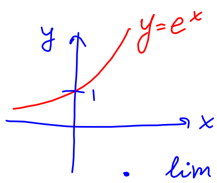

Then

$$\boxed{e \approx 2.7182818284590452}$$ $e>1$

$$\boxed{\frac{d}{dx}e^x = e^x}$$

- $\lim_{x\to\infty} e^x = \infty$
- $\lim_{x\to -\infty} e^x = 0$

The Chain Rule for $y=e^x$:

$$\boxed{\left(e^{u(x)}\right)' = u'(x)e^{u(x)}}$$

Example 2. Differentiate each function.

1. $f(x) = e^{\sqrt{x}}$

$$f'(x) = \left(e^{\sqrt{x}}\right)' = e^{\sqrt{x}}\left(\sqrt{x}\right)'$$
$$= e^{\sqrt{x}}\,\frac{1}{2}x^{-1/2} = \boxed{\frac{e^{\sqrt{x}}}{2\sqrt{x}}}$$

2. $f(x) = xe^{-x^2}$

Product Rule

$$f'(x) = \left(xe^{-x^2}\right)' = (x)'e^{-x^2} + x\left(e^{-x^2}\right)'$$
$$= e^{-x^2} + xe^{-x^2}(-x^2)'$$
$$= e^{-x^2} + xe^{-x^2}(-2x)$$
$$= e^{-x^2} - 2x^2e^{-x^2}$$
$$= \boxed{e^{-x^2}(1-2x^2)}$$

3. $f(x) = e^{x\tan x}$

$$f'(x) = e^{x\tan x}(x\tan x)' = e^{x\tan x}\Big(\overbrace{(x)'}^{1}\tan x + x\overbrace{(\tan x)'}^{\sec^2 x}\Big)$$
$$= \boxed{e^{x\tan x}\left(\tan x + x\sec^2 x\right)}$$

4. $f(x) = x^e$ Power function $(x^n)' = nx^{n-1}$

$$f'(x) = \boxed{ex^{e-1}}$$

Example 3. Show that the function $y = e^{2x} + e^{-3x}$ satisfy the differential equation

$$y'' + y' - 6y = 0$$

$$\boxed{y = e^{2x} + e^{-3x}}$$

$$y' = e^{2x}(2x)' + e^{-3x}(-3x)'$$

$$y' = \boxed{2e^{2x} - 3e^{-3x}}$$

$$y'' = 2(e^{2x})' - 3(e^{-3x})'$$

$$= 2(2e^{2x}) - 3(-3e^{-3x})$$

$$\boxed{y'' = 4e^{2x} + 9e^{-3x}}$$

Plug y, y', y'' into the equation $y'' + y' - 6y = 0$

$$\underbrace{(4e^{2x} + 9e^{-3x})}_{y''} + \underbrace{(2e^{2x} - 3e^{-3x})}_{y'} - 6\underbrace{(e^{2x} + e^{-3x})}_{y}$$

$$= 4e^{2x} + 9e^{-3x} + 2e^{2x} - 3e^{-3x} - 6e^{2x} - 6e^{-3x}$$

$$= e^{2x}(4+2-6) + e^{-3x}(9-3-6) = e^{2x}(0) + e^{-3x}(0) = 0$$