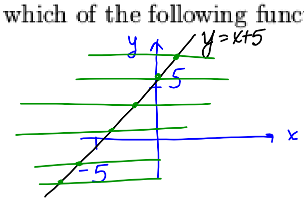
Section 4.2 **Inverse functions**

Definition. A function f with domain A is called one-to-one function if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Horizontal line test. A function is one-to-one if and only if no horizontal line intersects its graph more that once.

Example 1. Determine which of the following functions is one-to-one:

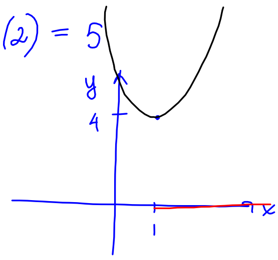
1. $f(x) = x + 5$



$\boxed{\text{one-to-one}}$

2. $g(x) = x^2 - 2x + 5$

$g(0) = 5$
$g(2) = 5$

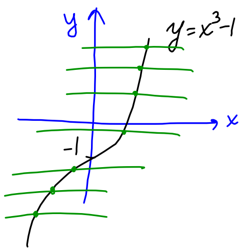


$0 \neq 2$
$g(0) = g(2)$

$\boxed{\text{not one-to-one on } (-\infty, \infty)}$

$\boxed{\text{one-to-one on } (1, \infty)}$

3. $h(x) = x^3 - 1$



$\boxed{\text{one-to-one}}$

4. $p(x) = x^4 + 5$

$p(-1) = (-1)^4 + 5 = 6$
$p(1) = 1^4 + 5 = 6$

$-1 \neq 1$
but $p(-1) = p(1)$

$\boxed{\text{not one-to-one}}$

Definition. Let f be one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Longleftrightarrow f(x) = y$$

for any y in B.

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

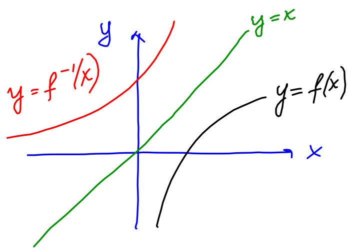
Let f be one-to-one function with domain A and range B. If $f(a) = b$, then $f^{-1}(b) = a$.

Cancellation equations

$$f^{-1}(f(x)) = x \text{ for every } x \in A$$
$$f(f^{-1}(x)) = x \text{ for every } x \in B$$

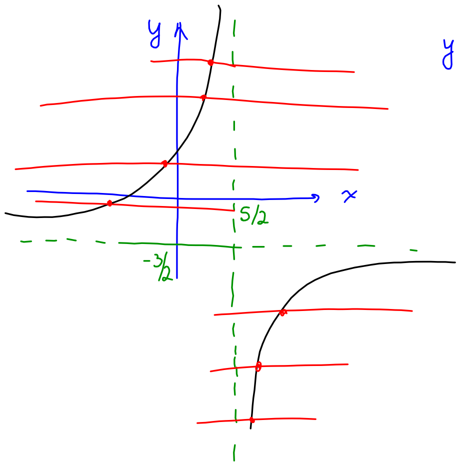
How to find the inverse function of a one-to-one function f

1. Write $y = f(x)$
2. Solve this equation for x in terms of y.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.



The graph of f^{-1} is obtained by the reflecting the graph f about the line $y = x$.

Example 2. Show that the function $f(x) = \dfrac{1+3x}{5-2x}$ is one-to-one and find $f^{-1}(y)$.



$y = \dfrac{1+3x}{5-2x}$

$\boxed{\text{one-to-one}}$

$$(5-2x)y = \frac{1+3x}{5-2x}(5-2x)$$

$$y(5-2x) = 1+3x$$

$$5y - 2xy = 1 + 3x$$

$$5y - 1 = 3x + 2xy$$

$$5y - 1 = x(3+2y)$$

$$x = \boxed{\frac{5y-1}{3+2y} = f^{-1}(y)}$$

Theorem. If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

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Theorem. If f is a one-to-one differentiable function with inverse function $\boxed{g = f^{-1}}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and

$$\boxed{g'(a) = \frac{1}{f'(g(a))}}$$

Example 3. Find $g'(4)$, where g is the inverse function of the function $f(x) = 3 + x + e^x$.

$$g'(4) = \frac{1}{f'(g(4))}$$

$g(4) - ?$ since $g = f^{-1}$

$g(4) = a \Leftrightarrow f(a) = 4$

$f(x) = 3 + x + e^x$

$f(0) = 3 + 0 + \underbrace{e^0}_{1} = 3 + 1 = 4$

$f(0) = 4 \Leftrightarrow g(4) = 0$

$$g'(4) = \frac{1}{f'(0)}$$

$f'(x) = 1 + e^x$

$f'(0) = 1 + e^0 = 2$

$$\boxed{g'(4) = \frac{1}{2}}$$

If $g(x)$ is the inverse function of $f(x)$, find $g'(3)$.

$$f(x) = \sqrt{2x^3 + 3x^2 + 3x + 1} = (2x^3+3x^2+3x+1)^{1/2}$$

$$g'(3) = \frac{1}{f'(g(3))}$$

Find a such that

$f(a)=3 \Rightarrow f(1)=3.$

$$2x^3+3x^2+3x+1=9$$

$$x=1$$

$$g(3)=1$$

$$g'(3) = \frac{1}{f'(1)}$$

$$f'(x) = \frac{1}{2}(2x^3+3x^2+3x+1)^{-1/2}(2x^3+3x^2+3x+1)'$$

$$= \frac{1}{2}(2x^3+3x^2+3x+1)^{-1/2}(6x^2+6x+3)$$

$$f'(1) = \frac{1}{2}\ \frac{1}{3}\ (6+6+3) = \frac{5}{2}$$

$$g'(3) = \frac{1}{f'(1)} = \boxed{\frac{2}{5}}$$