Section 4.3 **Logarithmic functions**

$y = \log_a x$ inverse for $y = a^x$, $a > 0$ a is a constant

$$\boxed{\log_a x = y \iff a^y = x}$$

The cancellation equations

$$\boxed{\log_a a^x = x \quad a^{\log_a x} = x}$$

If $a = e$, then

$$\log_e x = \ln x$$

$\log_{10} x = \log x$

Example 1. Evaluate:

1. $\log_2 64 = \log_2(2^6) = \boxed{6}$

   $\log_2(2^x) = x$

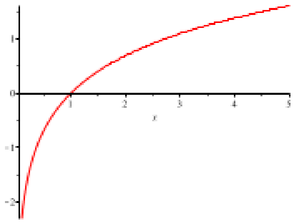
2. $\log_6 \dfrac{1}{36} = \log_6(6^{-2}) = \boxed{-2}$

3. $2^{\log_2 3 + \log_2 5} = 2^{\log_2 3} \cdot 2^{\log_2 5} = 3(5) = \boxed{15}$

   $2^{\log_2 x} = x$

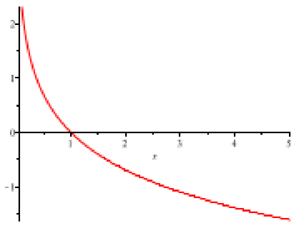
Theorem. Function $f(x) = \log_a x$ is one-to-one continuous function with domain $(0, \infty)$ and range $(-\infty, \infty)$.

1. If $a > 1$, then



- $f(x) = \log_a x$ is increasing function
- $\lim\limits_{x \to \infty} \log_a x = \infty$
- $\lim\limits_{x \to 0^+} \log_a x = -\infty$

2. If $0 < a < 1$, then



- $f(x) = \log_a x$ is decreasing function
- $\lim\limits_{x \to \infty} \log_a x = -\infty$
- $\lim\limits_{x \to 0^+} \log_a x = \infty$

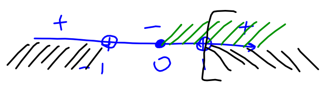
Example 2. Find the domain and the range for the function $f(x) = \sqrt{x} \ln(x^2 - 1)$.

$\sqrt{x} \Rightarrow x \geq 0$

$\ln(x^2 - 1) \Rightarrow x^2 - 1 > 0$

$(x-1)(x+1) > 0$

$(-\infty, -1) \cup (1, \infty)$



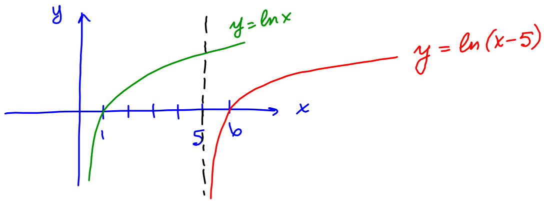
$0^2 - 1 = -1 < 0$

$\boxed{(1, \infty) \text{ DOMAIN}}$

$\boxed{(-\infty, \infty) \text{ RANGE}}$

Example 3. Find the limit:

1. $\lim_{x\to 5^+} \ln(x-5)$ $= -\infty$



2. $\lim_{x\to\infty} \log_2(x^2-x)$ $= \infty$

   $2>1$

3. $\lim_{x\to\infty} \dfrac{\ln x}{1+\ln x}$

   $\lim_{x\to\infty} \ln x = \infty$, substitution $t = \ln x$, $t\to\infty$ as $x\to\infty$

   $= \lim_{t\to\infty} \dfrac{t}{1+t} = \lim_{t\to\infty} \dfrac{\cancel{t}}{\cancel{t}(\frac{1}{t}+1)} = \lim_{t\to\infty} \dfrac{1}{\underbrace{\frac{1}{t}}_{0}+1} = \boxed{1}$

Properties of logatithmic functions

If $x, y > 0$ and k is a constant, then

a, b are positive constants

1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^k = k \, \log_a x \qquad \log_a \dfrac{1}{x} = -\log_a x$
4. $\log_{a^k} x = \dfrac{1}{k} \log_a x \qquad \log_{\frac{1}{a}} x = -\log_a x$
5. $\log_a a = 1$
6. $\log_a 1 = 0$
7. $\log_a x = \dfrac{\log_b x}{\log_b a}$

Example 4. Evaluate $e^{3\ln 2-1}\ln(5e^2)$

$= e^{3\ln 2}\cdot e^{-1}\left(\ln 5 + \ln(e^{2})\right)$

$= (e^{\ln 2})^3 e^{-1}\left(\ln 5 + 2\cancel{\ln e}\right)$

$= 2^3 e^{-1}(\ln 5 + 2) = \boxed{\frac{8}{e}(\ln 5+2)}$

$e^{3\ln 2} = (e^3)^{\ln 2}$

$= e^{\ln(2^3)} = 2^3$

$(e^{\ln 2})^3 = 2^3$

Example 5. Express the given quatities as a single logarithm:

1. $\log_2 x + 5\log_2(x+1) + \frac{1}{2}\log_2(x-1)$

$= \log_2 x + \log_2 (x+1)^5 + \log_2 (x-1)^{1/2}$

$= \boxed{\log_2 \left[x(x+1)^5(x-1)^{1/2}\right]}$

2. $2\ln 4 - \ln 2$ $= 2\ln(2^2) - \ln 2 = 4\ln 2 - \ln 2 = 3\ln 2 = \ln(2^3) = \boxed{\ln 8}$

$\ln(4^2) - \ln 2 = \ln 16 - \ln 2 = \ln\frac{16}{2} = \boxed{\ln 8}$

3. $\log_8 a - \log_4 b + \log_2 c$ $= \log_{2^3} a - \log_{2^2} b + \log_2 c$

$= \frac{1}{3}\log_2 a - \frac{1}{2}\log_2 b + \log_2 c$

$= \log_2 a^{1/3} - \log_2 b^{1/2} + \log_2 c$

$= \boxed{\log_2 \frac{a^{1/3}c}{b^{1/2}}}$

E.6 Solve the equation.

(a) $\log_{2} x = 4$

$x = 2^4 = \boxed{16}$

(b) $\log_3 x = -1$

$x = 3^{-1} = \boxed{\frac{1}{3}}$

(c) $\ln(x-4) = 1$

$x - 4 = e^1$

$x - 4 = e$

$\boxed{x = e + 4}$

(d) $\ln(x-1) + \ln(x-2) = 0$

domain
$x - 1 > 0,\ x > 1$
$x - 2 > 0,\ x > 2$
$\boxed{x > 2}$

$\ln(x-1)(x-2) = 0$

$(x-1)(x-2) = e^0$

$(x-1)(x-2) = 1$

$x^2 - 3x + 2 = 1$

$x^2 - 3x + 1 = 0$

$x_1 = \dfrac{3 + \sqrt{3^2 - 4}}{2} = \boxed{\dfrac{3+\sqrt{5}}{2}} > 2$

$x_2 = \dfrac{3 - \sqrt{3^2 - 4}}{2} = \dfrac{3 - \sqrt{5}}{2} < 2$ not valid

(e) $3^x = 4$

$\log_3(3^x) = \log_3 4$

$\boxed{x = \log_3 4}$

(f) $5^x = \dfrac{1}{125} = 5^{-3}$

$5^x = 5^{-3}$

$\boxed{x = -3}$

(g) $8^x = 16$

$(2^3)^x = 2^4$

$2^{3x} = 2^4 \Rightarrow 3x = 4$

$\boxed{x = \frac{4}{3}}$

Example 6. Solve the equation:

1. $\frac{10(1+e^{-x})^{-1}}{10} = \frac{3}{10}$

$(1+e^{-x})^{-1} = \frac{3}{10}$

$\frac{1}{1+e^{-x}} = \frac{3}{10}$

$1+e^{-x} = \frac{10}{3}$

$e^{-x} = \frac{10}{3} - 1 = \frac{7}{3}$

$e^{-x} = \frac{7}{3}$

$\ln(e^{-x}) = \ln\frac{7}{3}$

$-x = \ln\frac{7}{3}$

$x = -\ln\frac{7}{3}$

$\boxed{x = \ln\frac{3}{7}}$

2. $\log_2(2x+1) = 2 - \log_2(4x)$

$\log_2(2x+1) + \log_2(4x) = 2$

$\log_2 4x(2x+1) = 2$

$4x(2x+1) = 2^2$

$\frac{4x(2x+1)}{4} = \frac{4}{4}$

$x(2x+1) = 1$

$2x^2 + x - 1 = 0$

$x_1 = \frac{-1+\sqrt{1+2(4)}}{4} = \boxed{\frac{1}{2}}$

$x_2 = \frac{-1-\sqrt{1+8}}{4} = -1 < 0$

not valid

Domain

$2x+1>0 \rightarrow x>-\frac{1}{2}$

$4x>0 \rightarrow x>0$

E.7. Find the inverse function.

(a) $y = \ln(x+6)$

$$x+6 = e^y$$

$$x = e^y - 6 = f^{-1}(y)$$

$$\boxed{f^{-1}(x) = e^x - 6}$$

(b) $y = 2^{8^x}$

$$\log_2 y = \log_2 \left(2^{8^x}\right)$$

$$\log_2 y = 8^x \cancel{\log_2 2}\ 1$$

$$\log_2 y = 8^x$$

$$\log_8(\log_2 y) = \log_8 (8^x)$$

$$\log_8(\log_2 y) = x \cancel{\log_8 8}\ 1$$

$$x = \log_8(\log_2 y) = f^{-1}(y)$$

$$\boxed{f^{-1}(x) = \log_8(\log_2 x)}$$