Section 4.4 **Derivatives of logarithmic functions**

$$\boxed{(\ln x)' = \frac{1}{x}}$$

$$\boxed{(\ln(g(x)))' = \frac{g'(x)}{g(x)}}$$

Example 1. Differentiate each function

1. $f(x) = \sin x \ln x$

$$f'(x) = (\sin x)' \ln x + \sin x (\ln x)'$$
$$= \boxed{\cos x \ \ln x + \sin x \ \frac{1}{x}}$$

2. $f(x) = \ln(x^2 + 3x - 1)$

$$f'(x) = \frac{1}{x^2+3x-1} \ (x^2+3x-1)'$$
$$= \boxed{\frac{2x+3}{x^2+3x-1}}$$

3. $f(x) = \dfrac{1 - \ln x}{1 + \ln x}$

$$f'(x) = \frac{(1-\ln x)'(1+\ln x) - (1+\ln x)'(1-\ln x)}{(1+\ln x)^2} = \frac{-\frac{1}{x}(1+\ln x) - \frac{1}{x}(1-\ln x)}{(1+\ln x)^2}$$
$$= \frac{-\frac{1}{x} - \cancel{\frac{\ln x}{x}} - \frac{1}{x} + \cancel{\frac{\ln x}{x}}}{(1+\ln x)^2} = \boxed{-\frac{2}{x(1+\ln x)^2}}$$

4. $f(x) = \ln |x|$

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}, \qquad \ln|x| = \begin{cases} \ln x, & \text{if } x > 0 \\ \ln(-x), & \text{if } x < 0 \end{cases}$$

$$(\ln|x|)' = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \frac{1}{-x}(-x)', & \text{if } x < 0 \end{cases} = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \frac{-1}{-x}, & \text{if } x < 0 \end{cases} = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \frac{1}{x}, & \text{if } x < 0 \end{cases}$$

$$\boxed{(\ln|x|)' = \frac{1}{x}, \quad x \neq 0}$$

Example 2. Find the equation of the tangent line to the curve $x = \ln t$, $y = te^t$ at the point (0,e).

Find t such that $\begin{cases} \ln t = 0 \\ te^t = e \end{cases} \Rightarrow t = 1$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(te^t)'}{(\ln t)'} = \frac{e^t + te^t}{\frac{1}{t}} = t(e^t + te^t)$$
$$= te^t(1+t)$$

$$\left.\frac{dy}{dx}\right|_{t=1} = (1)e^1(1+1) = 2e$$

$$y = 2e(x-0) + e$$

$$\boxed{y = 2ex + e}$$

$y = f'(a)(x-a) + f(a)$

$f'(a) = 2e$, $a = 0$, $f(a) = e$.

Example 3. Find y' if $\frac{d}{dx} y = \frac{d}{dx} \ln(x^2 + y^2)$ Implicit Differentiation.

$$y' = \frac{1}{x^2+y^2} \frac{d}{dx}(x^2+y^2)$$

$$\left[y' = \frac{1}{x^2+y^2}(2x+2yy') \right] (x^2+y^2)$$

$$(x^2+y^2)y' = 2x + 2yy'$$

$$(x^2+y^2)y' - 2yy' = 2x$$

$$y'(x^2+y^2-2y) = 2x$$

$$\boxed{y' = \frac{2x}{x^2+y^2-2y}}$$

$$\boxed{(\log_a x)' = \frac{1}{x \ln a}}$$

$$\left[\log_a (g(x))\right]' = \frac{g'(x)}{g(x) \ln a}$$

Example 4. Find $\frac{d}{dx}\left(\log_3(\tan(x^2))\right)$

$$= \frac{1}{\tan(x^2) \ln 3} \left(\tan(x^2)\right)'$$

$$= \boxed{\frac{2x \sec^2(x^2)}{\tan(x^2) \ln 3}}$$

$$\boxed{(a^x)' = a^x \ln a}$$

$$\boxed{\left[a^{g(x)}\right]' = a^{g(x)} \ln a \; g'(x)}$$

Example 5. Find $\frac{d}{dx}\left(\sqrt{2 - 3^x} + \pi^{-x} + x^e\right)$

$$= \frac{1}{2}(2-3^x)^{-1/2} (2-3^x)' + \pi^{-x} \ln \pi \, (-x)' + e x^{e-1}$$

$$= \boxed{\frac{1}{2}(2-3^x)^{-1/2} \left(-3^x \ln 3\right) - \pi^{-x} \ln \pi + e x^{e-1}}$$

Steps in logarithmic differentiation

1. Take the logarithm of both sides of an equation.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

Example 6. Differentiate each function

1. $\ln y = \ln(x^x)$

$\ln y = \ln(x^x)$

$\frac{d}{dx} \ln y = \frac{d}{dx} x \ln x$

$\frac{y'}{y} = (x)' \ln x + x(\ln x)'$

$= \ln x + x \frac{1}{x} = \ln x + 1$

$\left(\frac{y'}{y} = \ln x + 1\right) y$

$y' = \underbrace{y}_{x^x}(\ln x + 1)$

$\boxed{y' = x^x(\ln x + 1)}$

2. $y = x^{\sin x}$

$\ln y = \ln(x^{\sin x})$

$\frac{d}{dx} \ln y = \frac{d}{dx} \sin x \ln x$

$\frac{y'}{y} = \cos x \ln x + \sin x \frac{1}{x}$

$y' = y\left(\cos x \ln x + \frac{\sin x}{x}\right)$

$\boxed{y' = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)}$

3. $y = \cos(\overset{z}{x^{\sqrt{x}}})$ $\qquad z = x^{\sqrt{x}}$

$y' = -\sin z \cdot z'$

$\ln z = \ln(x^{\sqrt{x}})$

$\frac{d}{dx} \ln z = \frac{d}{dx} \sqrt{x} \ln x$

$\frac{z'}{z} = \frac{1}{2} x^{-1/2} \ln x + \frac{\sqrt{x}}{x}$

$= \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$

3

$\frac{z'}{z} = \frac{\ln x + 2}{2\sqrt{x}}$

$z' = z \frac{\ln x + 2}{2\sqrt{x}}$

$z' = x^{\sqrt{x}} \frac{\ln x + 2}{2\sqrt{x}}$

$\boxed{y' = -\sin(x^{\sqrt{x}}) \cdot x^{\sqrt{x}} \frac{2 + \ln x}{2\sqrt{x}}}$

4. $\ln y = \ln \frac{(x+1)^4 \left[\sqrt[5]{x^2+1}\right]}{(x^3-1)^{10}(1+3x^2)^{2016}}$

$\ln y = \ln\left[(x+1)^4 \sqrt[5]{x^2+1}\right] - \ln\left[(x^3-1)^{10}(1+3x^2)^{2016}\right]$

$= \ln(x+1)^4 + \ln\left[\sqrt[5]{x^2+1}\right] - \ln(x^3-1)^{10} - \ln(1+3x^2)^{2016}$

$\frac{d}{dx} \ln y = \frac{d}{dx}\left(4 \ln(x+1) + \frac{1}{5} \ln(x^2+1) - 10 \ln(x^3-1) - 2016 \ln(1+3x^2)\right)$

$\frac{y'}{y} = \frac{4}{x+1} + \frac{1}{5} \cdot \frac{1}{x^2+1}(x^2+1)' - 10 \frac{1}{x^3-1}(x^3-1)' - 2016 \frac{1}{1+3x^2}(1+3x^2)'$

$\frac{y'}{y} = \frac{4}{x+1} + \frac{2x}{5(x^2+1)} - \frac{30x^2}{x^3-1} - \frac{12096x}{1+3x^2}$

$y' = y\left(\frac{4}{x+1} + \frac{2x}{5(x^2+1)} - \frac{30x^2}{x^3-1} - \frac{12096x}{1+3x^2}\right)$

$\boxed{y' = \frac{(x+1)^4 \sqrt[5]{x^2+1}}{(x^3-1)^{10}(1+3x^2)^{2016}}\left(\frac{4}{x+1} + \frac{2x}{5(x^2+1)} - \frac{30x^2}{x^3-1} - \frac{12096x}{1+3x^2}\right)}$

$$\boxed{\lim_{x \to 0}(1+x)^{1/x} = e}$$

$$\boxed{\lim_{x \to \infty}\left(1+\frac{1}{x}\right)^{x} = e}$$