Section 4.5 **Exponential growth and decay**

If $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to $y(t)$ at any time, then

$$\frac{dy}{dt} = ky$$

where k is a constant. This equation is called the **law of natural growth** if $k > 0$ or the ~~the~~ **law of natural decay** if $k < 0$.

The only solution to this equation is

$$\boxed{y(t) = y(0)e^{kt}}$$

Example 1. A bacteria culture starts with 500 bacteria and after 3 hours there are 8000 bacteria.

1. Find an expression for the number of bacteria after t hours.

$p(t)$ is the population after t hours.

$p(0) = 500$, $p(3) = 8000$

$p(t) = p(0)e^{kt} = 500e^{kt}$, k is an unknown constant

$p(3) = 500e^{3k} = 8000$

$e^{3k} = \frac{80}{5} = 16$

$3k = \ln 16$

$k = \frac{\ln 16}{3}$

$p(t) = 500\,e^{\frac{\ln 16}{3}t} = 500\left(\cancel{e^{\ln 16}}\,16\right)^{\frac{t}{3}}$

$\boxed{p(t) = 500\left(16^{t/3}\right)}$

2. Find the number of bacteria after 4 hours.

$p(4) = 500\left(16^{4/3}\right) \approx \boxed{20,158.74}$

3. When will the population reach 30,000?

Find t such that $p(t) = 30000$

$\frac{500(16^{t/3})}{500} = \frac{30000}{500}$

$16^{t/3} = 60$

$\ln(16^{t/3}) = \ln 60$

$\frac{t}{3}\ln 16 = \ln 60$

$\frac{t}{3} = \frac{\ln 60}{\ln 16}$

$t = 3\frac{\ln 60}{\ln 16}$

$\approx \boxed{4.43 \text{ hours}}$

Example 2. Polonium-214 has a half-life of 1.4×10^{-4} s.

1. If a sample has a mass of 50 mg, find a formula for the mass that remains after t seconds.

$m(t)$ is the mass after t sec.

$m(0) = 50, \quad m(t) = m(0)e^{kt}$

$m(t) = 50e^{kt}$

$m(1.4\times10^{-4}) = \frac{1}{2}m(0)$

$25 = 50e^{k(1.4\times10^{-4})}$

$\ln e^{k(1.4\times10^{-4})} = \ln\frac{1}{2}$

$(1.4\times10^{-4})k = \ln\frac{1}{2} = -\ln 2$

$k = -\frac{\ln 2}{1.4\times10^{-4}} = -\frac{\ln 2}{1.4}\cdot 10^4$

$m(t) = 50e^{-\frac{\ln 2}{1.4}10^4 t}$

$= 50\left(\cancel{e^{\ln 2}}\right)^{-\frac{10^4}{1.4}t}$ 2

$= \boxed{50\left(2^{-\frac{10^4}{1.4}t}\right)}$

2. Find the mass that remains after a hundredth of a second.

$m(0.01) = 50\left(2^{-\frac{10^4}{1.4}\frac{1}{100}}\right) = 50\left(2^{-\frac{100}{1.4}}\right) \approx \boxed{1.57\times10^{-20}\text{(mg)}}$

3. How long would it take for the mass to decay to 40 mg?

$m(t) = 40$

$40 = 50\left(2^{-\frac{10^5}{14}t}\right)$

$\ln\frac{4}{5} = \ln\left(2^{-\frac{10^5}{14}t}\right)$

$-\frac{10^5}{14}t = \frac{\ln\frac{4}{5}}{\ln 2}$

$t = -\frac{14}{10^5}\frac{\ln(4/5)}{\ln 2} \approx \frac{4.5}{10^5} = \boxed{4.5\times10^{-5}\text{ sec}}$

Example 3. A roast turkey is taken from the oven when its temperature has reached 185^0F and is placed on a table in a room where the temperature is 75^0F.

1. If the temperature of turkey is 150^0F after half an hour, what is the temperature after 45 min?

$T(t)$ is the temperature after t min, $T(0)=185$

Newton's Law of Cooling: $\frac{dT}{dt}=k(T-75)$

Substitution: $u=T-75$, $u'(t)=T'(t)-0=T'(t)$

$$\frac{du}{dt}=ku, \quad u(0)=T(0)-75=185-75=110$$

$$u(t)=u(0)e^{kt}=110e^{kt}$$

$$T(t)-75=110e^{kt}$$

$$T(t)=75+110e^{kt}, \quad T(30)=150$$

$$T(30)=\underbrace{75+110e^{30k}=150}$$

$$110e^{30k}=75, \ \ln(e^{30k})=\frac{75}{110}\ \ln\left(\frac{15}{22}\right)$$

$$30k=\ln\frac{15}{22}$$

$$k=\frac{1}{30}\ln\frac{15}{22}$$

$e^{\ln\frac{15}{22}}=\frac{15}{22}$

$$T(t)=75+110e^{\frac{t}{30}\ln\frac{15}{22}}=\boxed{75+110\left(\frac{15}{22}\right)^{\frac{t}{30}}=T(t)}$$

$$T(45)=75+110\left(\frac{15}{22}\right)^{\frac{45}{30}}\approx\boxed{137^\circ F}$$

2. When will the turkey have cooled to 100^0F ?

Find t such that $T(t)=100$

$$75+110\left(\frac{15}{22}\right)^{\frac{t}{30}}=100$$

$$110\left(\frac{15}{22}\right)^{\frac{t}{30}}=25$$

$$\ln\left(\frac{15}{22}\right)^{\frac{t}{30}}=\frac{25}{110}=\ln\frac{5}{22}$$

$$\frac{t}{30}\ln\left(\frac{15}{22}\right)=\ln\frac{5}{22}$$

$$t=30\frac{\ln\frac{5}{22}}{\ln\frac{15}{22}}\approx\boxed{116\text{ min}}$$

Example 4. A tank contains 1500 L of brine with a concentration of 0.3 kg of salt per liter. In order to dilute the solution, pure water is run into the tank at a rate of 20 L/min and the resulting solution, which is stirred continuously, runs out at the same rate.

1. How many kilograms of salt will remain after 30 min?

$q(t)$ is amount of salt in the tank after t min

$q(0) = (1500)(0.3) = 450$ kg

$\frac{dq}{dt} = [\text{rate in}] - [\text{rate out}]$

$[\text{rate in}] = 0$

concentration of salt at time t.

$[\text{rate out}] = \frac{q(t)}{1500} \cdot (20) = \frac{q(t)}{75}$

$\frac{dq}{dt} = 0 - \frac{q(t)}{75}$ or $\frac{dq}{dt} = \underbrace{\left(-\frac{1}{75}\right)}_{=k} \cdot q(t)$

$q(t) = q(0)e^{-\frac{t}{75}}$

$\boxed{q(t) = 450e^{-\frac{t}{75}}}$

$q(30) = 450e^{-\frac{30}{75}} \approx \boxed{302 \text{ kg}}$

2. When will the concentration of salt be reduced to 0.2 kg/L?

$\frac{q(t)}{1500}$ concentration of salt after t min

Find t such that $\frac{q(t)}{1500} = 0.2$

$\frac{450}{1500} = \frac{45}{150} = \frac{3 \cdot 15}{10 \cdot 15}$

$\frac{45\cancel{0}}{150\cancel{0}} e^{-\frac{t}{75}} = 0.2$

$0.3e^{-\frac{t}{75}} = 0.2$

$\ln e^{-\frac{t}{75}} = \ln\frac{2}{3}$

$-\frac{t}{75} = \ln\frac{2}{3}$ or $t = -75\ln\frac{2}{3} \approx \boxed{30 \text{ (min)}}$