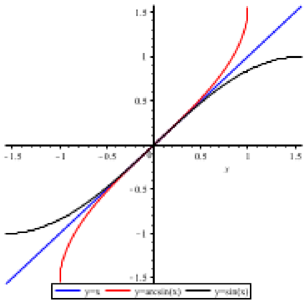
Section 4.6 **Inverse trigonometric functions**

Inverse sine function

$$\arcsin x = \sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$

DOMAIN $\quad -1 \leq x \leq 1$
RANGE $\quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



CANCELLATION EQUATIONS

$$\boxed{\sin^{-1}(\sin x) = x} \quad \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\boxed{\sin(\sin^{-1} x) = x} \quad \text{for } -1 \leq x \leq 1$$

$\sin^{-1} x \neq \frac{1}{\sin x}$

Example 1. Find

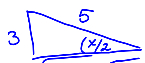
1. $\sin^{-1}(0.5) = \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$

   $\sin\frac{\pi}{6} = \frac{1}{2}$, so $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

2. $\sin^{-1}(\sin 1) = 1$

3. $\sin(\underbrace{2\sin^{-1}\tfrac{3}{5}}_{x}) = \sin x$, $\quad x = 2\sin^{-1}\frac{3}{5}$

   $\sin^{-1}\frac{3}{5} = \frac{x}{2}$ or $\sin\frac{x}{2} = \frac{3}{5}$



$\sqrt{25-9} = \sqrt{16} = 4$

$\cos\frac{x}{2} = \frac{4}{5}$

$\sin x = 2\cancelto{\frac{3}{5}}{\sin\frac{x}{2}}\,\cancelto{\frac{4}{5}}{\cos\frac{x}{2}} = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \boxed{\frac{24}{25}}$

4. $\arcsin(\sin\frac{5\pi}{4}) = \arcsin\left(-\frac{1}{\sqrt{2}}\right) = \boxed{-\frac{\pi}{4}}$

Example 2. Simplify $\tan(\underbrace{\sin^{-1}x}_{y}) = \tan y$, $y = \sin^{-1}x$

$x = \sin y$, $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

$$\tan y = \frac{\sin y}{\cos y} = \frac{\sin y}{\sqrt{1-\sin^2 y}} = \boxed{\frac{x}{\sqrt{1-x^2}}}$$

Find the derivative of $y = \sin^{-1}x$.

$\frac{d}{dx}x = \frac{d}{dx}\sin y$ Implicit Differentiation

$1 = \cos y \cdot y' \Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

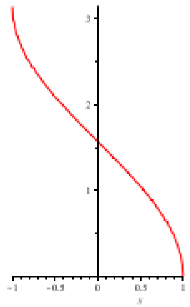
$$\boxed{(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}}$$

Inverse cosine function

$$\arccos x = \cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

DOMAIN $\quad -1 \le x \le 1$

RANGE $\quad 0 \le y \le \pi$



CANCELLATION EQUATIONS

$$\boxed{\cos^{-1}(\cos x) = x} \text{ for } 0 \le x \le \pi$$

$$\boxed{\cos(\cos^{-1} x) = x} \text{ for } -1 \le x \le 1$$

$$\boxed{(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}}$$

$\sqrt{1-x^2}$

Inverse tangent function

$$\arctan x = \tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

DOMAIN $\quad -\infty \leq x \leq \infty$

RANGE $\quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

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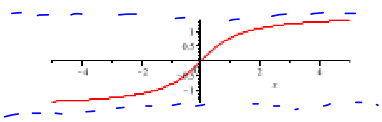


CANCELLATION EQUATIONS

$$\tan^{-1}(\tan x) = x \quad \text{for} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\tan(\tan^{-1} x) = x \quad \text{for} \quad -\infty \leq x \leq \infty$$

$$\boxed{\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}}$$

$$\boxed{\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}}$$

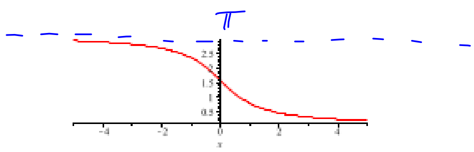
$$\boxed{(\tan^{-1} x)' = \frac{1}{1+x^2}}$$

Inverse cotangent function

$$\mathrm{arccot} x = \cot^{-1} x = y \quad \Leftrightarrow \quad \cot y = x$$

DOMAIN $\quad -\infty \leq x \leq \infty$

RANGE $\quad 0 < y < \pi$



CANCELLATION EQUATIONS

$$\cot^{-1}(\cot x) = x \quad \text{for } 0 \leq x \leq \pi$$

$$\cot(\cot^{-1} x) = x \quad \text{for } -\infty \leq x \leq \infty$$

$$\lim_{x \to -\infty} \cot^{-1} x = \pi$$

$$\lim_{x \to \infty} \cot^{-1} x = 0$$

$$(\cot^{-1} x)' = -\frac{1}{1 + x^2}$$

Other inverse trigonometric functions

$$\csc^{-1} x = y \quad \Leftrightarrow \quad \csc y = x$$

DOMAIN $\quad |x| \geq 1$

RANGE $\quad y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$

$$\boxed{(\csc^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}}}$$

$$\sec^{-1} x = y \quad \Leftrightarrow \quad \sec y = x$$

DOMAIN $\quad |x| \geq 1$
RANGE $\quad y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

$$\boxed{(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}}$$

Example 3. Differentiate each function:

$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$

1. $f(x) = \sin^{-1}(2x-1)$

$$f'(x) = \frac{1}{\sqrt{1-(2x-1)^2}}(2x-1)' = \boxed{\frac{2}{\sqrt{1-(2x-1)^2}}}$$

$(\cos^{-1}x)' = -\frac{1}{\sqrt{1-x^2}}$

2. $g(x) = x\cos^{-1}x - \sqrt{1-x^2}$

$$g'(x) = (x)'\cos^{-1}x + x(\cos^{-1}x)' - \frac{1}{2}(1-x^2)^{-1/2}(1-x^2)'$$

$$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} + \frac{1}{\cancel{2}}(1-x^2)^{-1/2}(\cancel{+2}x)$$

$$= \boxed{\cos^{-1}x}$$

$(\tan^{-1}x)' = \frac{1}{1+x^2}$

3. $h(x) = \sin^{-1}(\tan^{-1}x)$

$$h'(x) = \frac{1}{\sqrt{1-(\tan^{-1}x)^2}}(\tan^{-1}x)' = \boxed{\frac{1}{\sqrt{1-(\tan^{-1}x)^2}} \frac{1}{1+x^2}}$$

$(a^{g(x)})' = a^{g(x)} \ln a \, g'(x)$

$(\csc^{-1}t)' = -\frac{1}{t\sqrt{t^2-1}}$

4. $u(t) = 2^{\csc^{-1}t}$

$$u'(t) = 2^{\csc^{-1}t}(\ln 2)(\csc^{-1}t)'$$

$$= \boxed{2^{\csc^{-1}t}(\ln 2)\left(-\frac{1}{t\sqrt{t^2-1}}\right)}$$