## Section 4.6 Inverse trigonometric functions

## Inverse sine function

$$
\arcsin x=\sin ^{-1} x=y \Leftrightarrow \sin y=x
$$

DOMAIN $\quad-1 \leq x \leq 1$
RANGE $\quad-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$


CANCELLATION EQUATIONS

$$
\begin{array}{|c|c}
\hline \sin ^{-1}(\sin x)=x & \text { for }-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
\hline \sin \left(\sin ^{-1} x\right)=x & \text { for }-1 \leq x \leq 1
\end{array}
$$

$$
\sin ^{-1} x \neq \frac{1}{\sin x}
$$

Example 1. Find

1. $\sin ^{-1}(0.5)=\sin ^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

$$
\sin \frac{\pi}{6}=\frac{1}{2} \text {, so } \sin ^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}
$$

2. $\sin ^{-1}(\sin 1)=1$
3. $\sin (\underbrace{2 \sin ^{-1} \frac{3}{5}}_{x})=\sin x, \quad x=2 \sin ^{-1} \frac{3}{5}$

$$
\sin ^{-1} \frac{3}{5}=\frac{x}{2} \text { or } \sin \frac{x}{2}=\frac{3}{5}
$$

$$
3 \underbrace{\sqrt{25-9}=\sqrt{16}}_{\sqrt{\frac{4}{2} / 2}}=4
$$

$$
\sin x=2 \sin \frac{x^{\frac{3}{5}}}{2} \cos \frac{x}{2}^{\frac{4}{5}}=2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)=\frac{24}{25}
$$

4. $\arcsin \left(\sin \frac{5 \pi}{4}\right)=\arcsin \left(-\frac{1}{\sqrt{2}}\right)=-\frac{\pi}{4}$

Example 2. Simplify $\tan (\overbrace{\sin ^{-1} x}^{y}=\tan y, \quad \begin{array}{l}y=\sin ^{-1} x \\ x=\sin y, \quad \cos y=\sqrt{1-\sin ^{2} y}=\sqrt{1-x^{2}}\end{array}$ $\tan y=\frac{\sin y}{\cos y}=\frac{\sin y}{\sqrt{1-\sin ^{2} y}}=\frac{x}{\sqrt{1-x^{2}}}$

Find the derivative of $\begin{aligned} y & =\sin ^{-1} x . ~ I m p l i c i t ~ D i f f e r e n t i a t i o n ~\end{aligned}$ $d x d x$
$1=\cos y \cdot y^{\prime} \Rightarrow y^{\prime}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin ^{2} y}}=\frac{1}{\sqrt{1-x^{2}}}$

$$
\left(\sin ^{-1} x\right)^{\prime}=\frac{1}{\sqrt{1-x^{2}}}
$$

Inverse cosine function

$$
\arccos x=\cos ^{-1} x=y \quad \Leftrightarrow \quad \cos y=x
$$

DOMAIN $\quad-1 \leq x \leq 1$
RANGE $0 \leq y \leq \pi$


CANCELLATION EQUATIONS

$$
\begin{gathered}
\cos ^{-1}(\cos x)=x \text { for } 0 \leq x \leq \pi \\
\cos \left(\cos ^{-1} x\right)=x \quad \text { for }-1 \leq x \leq 1 \\
\left(\cos ^{-1} x\right)^{\prime}=-\frac{1}{\sqrt{1-x^{2}}}
\end{gathered}
$$

$$
\vee 1-x^{-}
$$

## Inverse tangent function

$$
\arctan x=\tan ^{-1} x=y \Leftrightarrow \tan y=x
$$

DOMAIN $\quad-\infty \leq x \leq \infty$
RANGE $\quad-\frac{\pi}{2}<y<\frac{\pi}{2}$


CANCELLATION EQUATIONS

$$
\begin{aligned}
& \tan ^{-1}(\tan x)=x \text { for }-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
& \tan \left(\tan ^{-1} x\right)=x \text { for }-\infty \leq x \leq \infty \\
& \hline \frac{\lim _{x \rightarrow-\infty} \tan ^{-1} x=-\frac{\pi}{2}}{\lim _{x \rightarrow \infty} \tan ^{-1} x=\frac{\pi}{2}} \\
& \hline\left(\tan ^{-1} x\right)^{\prime}=\frac{1}{1+x^{2}}
\end{aligned}
$$

Inverse cotangent function

$$
\operatorname{arccot} x=\cot ^{-1} x=y \Leftrightarrow \cot y=x
$$

DOMAIN $\quad-\infty \leq x \leq \infty$
RANGE $0<y<\pi$


CANCELLATION EQUATIONS

$$
\left.\begin{array}{c}
\cot ^{-1}(\cot x)=x \text { for } 0<x \leq \pi \\
\cot \left(\cot ^{-1} x\right)=x \text { for }-\infty \leq x \leq \infty \\
\frac{\lim _{x \rightarrow-\infty} \cot ^{-1} x=\boldsymbol{\Pi}}{\lim _{x \rightarrow \infty} \cot ^{-1} x=\boldsymbol{0}} \\
\left(\cot ^{-1} x\right)^{\prime}=-\frac{1}{1+x^{2}}
\end{array}\right] .
$$

## Other inverse trigonometric functions

$$
\csc ^{-1} x=y \Leftrightarrow \csc y=x
$$

DOMAIN $\quad|x| \geq 1$

RANGE $\quad y \in\left(0, \frac{\pi}{2}\right] \cup\left(\pi, \frac{3 \pi}{2}\right]$

$$
\begin{aligned}
& \left(\csc ^{-1} x\right)^{\prime}=-\frac{1}{x \sqrt{x^{2}-1}} \\
& \sec ^{-1} x=y \Leftrightarrow \sec y=x
\end{aligned}
$$

DOMAIN $\quad|x| \geq 1$
RANGE $\quad y \in\left[0, \frac{\pi}{2}\right) \cup\left[\pi, \frac{3 \pi}{2}\right)$

$$
\left(\sec ^{-1} x\right)^{\prime}=\frac{1}{x \sqrt{x^{2}-1}}
$$

Example 3. Differentiate each function:

$$
\begin{array}{r}
\left(\sin ^{-1} x\right)^{\prime}=\frac{1}{\sqrt{1-x^{2}}} \cdot f(x)=\sin ^{-1}(2 x-1) \\
f^{\prime}(x)=\frac{1}{\sqrt{1-(2 x-1)^{2}}} \quad(2 x-1)^{\prime}=\frac{2}{\sqrt{1-(2 x-1)^{2}}}
\end{array}
$$

$$
\begin{aligned}
&\left(\cos ^{-1} x\right)^{\prime}=\frac{-12}{\sqrt{1-x^{2}}} g(x)=x \cos ^{-1} x-\sqrt{1-x^{2}} \\
& g^{\prime}(x)=(x)^{\prime} \cos ^{-1} x+x\left(\cos ^{-1} x\right)^{\prime}-\frac{1}{2}\left(1-x^{2}\right)^{-1 / 2}\left(1-x^{2}\right)^{\prime} \\
&=\cos ^{-1} x-\frac{x}{\sqrt{1-x^{2}}}+\frac{1}{\not x}\left(1-x^{2}\right)^{-1 / 2}(+\not 2 x) \\
&=\cos ^{-1} x
\end{aligned}
$$

$$
\begin{aligned}
\left(\tan ^{-1} x\right)^{\prime} & =\frac{13}{1+x^{2}} \quad h(x)=\sin ^{-1}\left(\tan ^{-1} x\right) \\
h^{\prime}(x) & =\frac{1}{\sqrt{1-\left(\tan ^{-1} x\right)^{2}}}\left(\tan ^{-1} x\right)^{\prime}=\frac{1}{\sqrt{1-\left(\tan ^{-1} x\right)^{2}}} \frac{1}{1+x^{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \left(a^{g(x)}\right)^{\prime}=a^{g(x)} \ln a g(x) u(t)=2^{\csc ^{-1} t} \\
& \left(\operatorname{cx}^{-1} t\right)^{\prime}=-\frac{1}{t \sqrt{t^{2}-1}}
\end{aligned}
$$

$$
\begin{aligned}
u^{\prime}(t) & =2^{\operatorname{cx}^{-1} t}(\ln \alpha)\left(\operatorname{ses}^{-1} t\right)^{\prime} \\
& =2^{\operatorname{cxc}^{-1} t}(\ln 2)\left(-\frac{1}{t \sqrt{t^{2}-1}}\right)
\end{aligned}
$$

