Section 4.8 **Indeterminate forms and L'Hospitale's rule**

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ for points close to a (except, possibly a). Suppose that

$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Then

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right| = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

Example 1. Evaluate $\lim_{x\to 0} \frac{(x\cos x - \sin x)'}{(x^3)'} = \left|\frac{0}{0}\right| = \lim_{x\to 0} \frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{3x^2} = \lim_{x\to 0} -\frac{\cancel{x}\sin x}{3x^{\cancel{2}}}$

$$= -\lim_{x\to 0} \frac{(\sin x)'}{(3x)'} = \left|\frac{0}{0}\right| = -\lim_{x\to 0} \frac{\cos x}{3} = \boxed{-\frac{1}{3}}$$

Indeterminate products If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} f(x)g(x) = |\infty \cdot 0| = \lim_{x\to a} \frac{f(x)}{1/g(x)} = \lim_{x\to a} \frac{g(x)}{1/f(x)} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right|$$

and now we can use L'Hospital's Rule.

Example 2. Evaluate $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2} = |0 \cdot \infty| = \lim_{x\to 1} \frac{(1-x)'}{\left(\cot\frac{\pi x}{2}\right)'} \left|\frac{0}{0}\right|$

$$= \lim_{x\to 1} \frac{-1}{-\csc^2\frac{\pi x}{2}\left(\frac{\pi}{2}\right)} = \lim_{x\to 1} \frac{2}{\pi} \sin^2\frac{\pi x}{2} = \frac{2}{\pi}\sin^2\frac{\pi}{2} = \boxed{\frac{2}{\pi}}$$

Indeterminate differences If we have to find $\lim_{x\to a}(f(x) - g(x)) = |\infty - \infty|$ ($\lim_{x\to a} f(x) = \infty, \lim_{x\to a} g(x) = \infty$), then we have to convert the difference into a quotient (by using a common denominator or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right|$ and we can use L'Hospital's Rule.

Example 3. Evaluate $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$ $|\infty - \infty|$

$$= \lim_{x\to 1} \frac{(x\ln x - (x-1))'}{((x-1)\ln x)'} \left|\frac{0}{0}\right| = \lim_{x\to 1} \frac{\ln x + x\frac{1}{x} - 1}{\ln x + (x-1)\frac{1}{x}} = \lim_{x\to 1} \frac{\ln x + \cancel{1} - \cancel{1}}{\ln x + \frac{x-1}{x}}$$

$$= \lim_{x\to 1} \frac{\ln x}{\frac{x\ln x + (x-1)}{x}} = \lim_{x\to 1} \frac{(x\ln x)'}{(x\ln x + (x-1))'} \left|\frac{0}{0}\right|$$

$$= \lim_{x\to 1} \frac{\ln x + x\frac{1}{x}}{\ln x + x\frac{1}{x} + 1} = \lim_{x\to 1} \frac{\ln x + 1}{\ln x + 2} = \frac{\cancel{\ln 1}^{0} + 1}{\cancel{\ln 1}^{0} + 2} = \boxed{\frac{1}{2}}$$

Indeterminate powers $\lim_{x\to a}[f(x)]^{g(x)} = |0^0 \text{ or } \infty^0 \text{ or } 1^\infty| = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a}[g(x)\ln f(x)]}$.

Now let's find

$$\lim_{x\to a}[g(x)\ln f(x)] = |0\cdot\infty| = \lim_{x\to a}\frac{\ln f(x)}{\frac{1}{g(x)}} = \left|\tfrac{0}{0} \text{ or } \tfrac{\infty}{\infty}\right| = b$$

Then

$$\lim_{x\to a}[f(x)]^{g(x)} = e^b$$

Example 4. Evaluate $\lim_{x\to 0}(1+4x^2)^{\frac{1}{x}}$ $|1^\infty|$ $\lim_{x\to 0} e^{\frac{1}{x}\ln(1+4x^2)} = e^{\lim_{x\to 0}\frac{1}{x}\ln(1+4x^2)}$

$$\lim_{x\to 0}\frac{1}{x}\ln(1+4x^2) = \lim_{x\to 0}\frac{(\ln(1+4x^2))'}{(x)'} \quad \left|\frac{0}{0}\right|$$

$$= \lim_{x\to 0}\frac{\frac{1}{1+4x^2}(8x)}{1} = \lim_{x\to 0}\frac{8x}{1+4x^2} = 0$$

$$\lim_{x\to 0}(1+4x^2)^{\frac{1}{x}} = e^0 = \boxed{1}$$

Find the limit. Use l'Hospital's Rule if appropriate.

$$\lim_{x \to 0} \frac{\left(e^{5x} - 1 - 5x\right)'}{\left(x^2\right)'} = \left|\frac{0}{0}\right| = \lim_{x \to 0} \frac{5e^{5x} - 5}{2x} = \frac{5}{2} \lim_{x \to 0} \frac{\left(e^{5x} - 1\right)'}{(x)'} \left|\frac{0}{0}\right|$$

$$= \frac{5}{2} \lim_{x \to 0} \frac{5e^{5x}}{1} = \frac{5}{2} \cdot 5\cancel{e^{0}} = \boxed{\frac{25}{2}}$$

Evaluate the limit.

$$\lim_{x \to \infty} \frac{(\ln(\ln 6x))'}{(6x)'} \left|\frac{\infty}{\infty}\right| = \lim_{x\to\infty} \frac{\frac{1}{\ln 6x}(\ln 6x)'}{6} = \lim_{x\to\infty} \frac{\frac{1}{\ln 6x}\cdot\frac{1}{6x}\cdot\cancel{6}}{\cancel{6}}$$

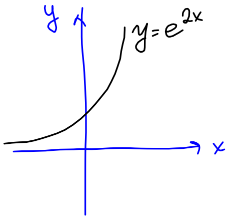
$$= \lim_{x\to\infty} \frac{1}{6x\ln 6x} = \frac{1}{\infty} = \boxed{0}$$

Evaluate the limit.

$e^{2x} = \frac{1}{e^{-2x}}$

$$\lim_{x \to -\infty} x^2 e^{2x} \quad \left| 0 \cdot \infty \right| = \lim_{x \to -\infty} \frac{(x^2)'}{(e^{-2x})'} = \left| \frac{\infty}{\infty} \right|$$

$$= \lim_{x \to -\infty} \frac{\cancel{2}x}{-\cancel{2}e^{-2x}} \quad \left| \frac{-\infty}{-\infty} \right| = \lim_{x \to -\infty} \frac{1}{2e^{-2x}} = \frac{1}{2} \lim_{x \to -\infty} \overset{0}{e^{2x}} = \boxed{0}$$



Find the limit. Use l'Hospital's Rule if appropriate. If there is a more elementary method, consider using it.

$$\lim_{x\to\infty} x^{7/x} \quad \left|\infty^0\right| = \lim_{x\to\infty} e^{\frac{7}{x}\ln x} = e^{\lim\limits_{x\to\infty} \frac{7}{x}\ln x} = e^0 = \boxed{1}$$

$$\lim_{x\to\infty} \frac{7}{x}\ln x = 7\lim_{x\to\infty} \frac{(\ln x)'}{(x)'} \quad \left|\frac{\infty}{\infty}\right| = 7\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 7\lim_{x\to\infty}\frac{1}{x} = 0$$