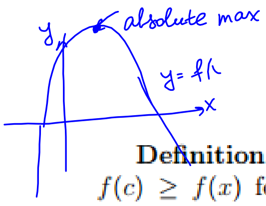
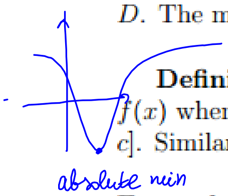
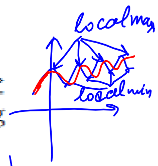


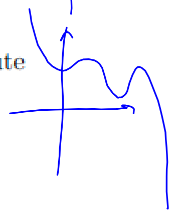
Section 5.2 **Maximum and minimum values**

Definition. A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** or **global minimum** at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and the minimum values of f are called the **extreme values** of f.

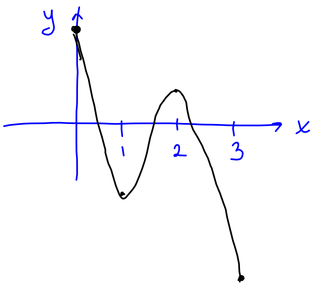




Definition. A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ when x is near c. [This means that $f(c) \geq f(x)$ for all x in some *open* interval containing c]. Similarly, f has a **local minimum** at c if $f(c) \leq f(x)$ when x is near c.



Example 1. Sketch the graph of the function f that is continuous on $[0, 3]$ and has the absolute maximum at 0, absolute minimum at 3, local minimum at 1, local maximum at 2.



The extreme value theorem. If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$

Definition. A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 2. Find the critical numbers of the function.

(a.) $f(x) = 4x^3 - 9x^2 - 12x + 3$

$$f'(x) = \frac{12x^2 - 18x - 12}{6} = \frac{0}{6}$$

$$2x^2 - 3x - 2 = 0$$

$$x_1 = \frac{3 + \sqrt{9 + 4(2)(2)}}{4} = \frac{3+5}{4} = \boxed{2}$$

$$x_2 = \frac{3-5}{4} = \boxed{-\frac{1}{2}}$$

(b.) $f(x) = \dfrac{x}{x^2 + 1}$

$$f'(x) = \frac{x^2 + 1 - 2x(x)}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2} = 0$$

$x^2 + 1$ is never zero.

$1 - x^2 = 0$ or $\boxed{x = \pm 1}$

If f has a local extremum at c, then c is a critical number of f.

The closed interval method. To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b)
2. Find $f(a)$ and $f(b)$
3. The largest number of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 3. Find the absolute maximum and absolute minimum values of f on the given interval.

(a.) $f(x) = x^2 + \frac{2}{x}$, $[1/2, 2]$

$f'(x) = 2x - \frac{2}{x^2} = \frac{2x^3 - 2}{x^2} = 0$ | The derivative DNE when $x \neq 0$ not in $(\frac{1}{2}, 2)$

$2x^3 - 2 = 0$ or $x^3 = 1$ or $x = 1$

$f(\frac{1}{2}) = (\frac{1}{2})^2 + \frac{2}{1/2} = \frac{1}{4} + 4 = \frac{17}{4}$

$f(1) = 1 + \frac{2}{1} = 3$ absolute min value

$f(2) = 2^2 + \frac{2}{2} = 5$ absolute max value (b.) $f(x) = \cos x + \sin x$, $[0, \pi/3]$

1. Find all critical numbers in $(0, \frac{\pi}{3})$.

$$f'(x) = -\sin x + \cos x = 0$$

$$\frac{\cos x}{\cos x} = \frac{\sin x}{\cos x}$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}$$

2. $f(0) = \cos 0 + \sin 0 = 1$ absolute min

$$f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \approx 1.4$$ absolute max

$$f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} + \sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}+1}{2} \approx 1.37$$

(c.) $f(x) = xe^{-x}$, $[0, 2]$

1. Find critical numbers on $(0,2)$

$f'(x) = (x)'e^{-x} + x(e^{-x})'$

$= e^{-x} + x(-e^{-x}) = e^{-x}(1-x) = 0$

$e^{-x} \neq 0$ $\quad 1-x=0$ or $x=1$

2. $f(0) = 0$ absolute min

$f(1) = 1e^{-1} = \frac{1}{e} \approx 0.37$ absolute max

$f(2) = 2e^{-2} = \frac{2}{e^2} \approx 0.27$