Section 5.3 **Derivatives and the shapes of curves.**

The mean value theorem If f is a differentiable function on the interval $[a, b]$, then there exist a number c, $a < c < b$, such that

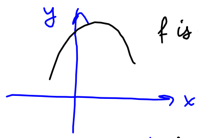
$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f(b) - f(a) = f'(c)(b - a)$$

Increasing/decreasing test

1. If $f'(x) > 0$ on an interval, then f is increasing on that interval
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval

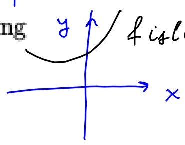
The first derivative test Suppose that c is a critical number of a continuous function f.

1. If f' changes from positive to negative at c, then f has a local max at c.
2. If f' changes from negative to positive at c, then f has a local min at c.
3. If f' does not change sign c, then f has a no local max or min at c.



f is CD

Definition. A function is called **concave upward** (CU) on an interval I if f' is an increasing function on I. It is called **concave downward** (CD) on I if f' is an decreasing on I.



f is CU

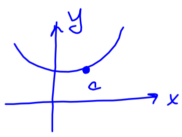
A point where a curve changes its direction of concavity is called an **inflection point**.

Concavity test

1. If $f''(x) > 0$ on an interval, then f is CU on this interval.
2. If $f''(x) < 0$ on an interval, then f is CD on this interval.

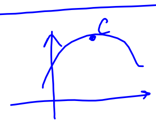
The second derivative test Suppose f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.



$\left.\begin{array}{l} f'(c) = 0 \\ f''(c) > 0 \end{array}\right\}$ local min

f is CU



$\left.\begin{array}{l} f'(c) = 0 \\ f''(c) < 0 \end{array}\right\}$ local max

f is CD

Example 1. Sketch the graph of the function

(a.) $f(x) = e^{-\frac{1}{x+1}}$

1. x-, y- intercept.

$e^{-\frac{1}{x+1}} \neq 0$ (never intercepts the x-axis)

y-intercept: $x=0$ $\quad f(0) = e^{-\frac{1}{0+1}} = e^{-1} \approx 0.37$ $\quad \boxed{(0, 0.37)}$ y-intercept

2. Asymptotes.

$\lim_{x \to -1^-} e^{-\frac{1}{x+1}} = \lim_{y \to \infty} e^{y} = \infty$ $\qquad$ $\lim_{x \to -1^+} e^{-\frac{1}{x+1}} = \lim_{y \to -\infty} e^{y} = 0$

$\lim_{x \to -1^-} \left(-\frac{1}{x+1}\right) = \infty$ $\qquad$ $\lim_{x \to -1^+} \left(-\frac{1}{x+1}\right) = -\infty$

$x = -1.01$

$\boxed{x=-1 \quad V.A.}$

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{-\frac{1}{x+1}} = e^{-\lim_{x \to \infty} \frac{1}{x+1}} = e^{0} = 1$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} e^{-\frac{1}{x+1}} = e^{0} = 1$

$\boxed{y=1 \quad H.A.}$

3. Continuity.

$x=-1$ infinite discontinuity.

continuous on $(-\infty, -1) \cup (-1, \infty)$

4. $f'(x) = \left(e^{-\frac{1}{x+1}}\right)' = e^{-\frac{1}{x+1}}\left(-\frac{1}{x+1}\right)'$

$= e^{-\frac{1}{x+1}}\left(-\left(-\frac{1}{(x+1)^2}\right)\right)$

$= \dfrac{e^{-\frac{1}{x+1}}}{(x+1)^2} > 0$, if $x \neq -1$

f is increasing on $(-\infty, -1) \cup (-1, \infty)$

no local min/local max.

5. $f''(x) = \left(\dfrac{e^{-\frac{1}{x+1}}}{(x+1)^2}\right)' = \dfrac{e^{-\frac{1}{x+1}}\left(+\frac{1}{\cancel{(x+1)^2}}\right)\cancel{(x+1)^2} - 2(x+1)\,e^{-\frac{1}{x+1}}}{(x+1)^4}$

$= \dfrac{e^{-\frac{1}{x+1}}\left(+1-2(x+1)\right)}{(x+1)^4} = \dfrac{e^{-\frac{1}{x+1}}(-2x-1)}{(x+1)^4} = 0$

$-2x-1=0$ or $x = -\frac{1}{2}$

$f\left(-\frac{1}{2}\right) = e^{-\frac{1}{-1/2+1}} = e^{-2} \approx 0.135$

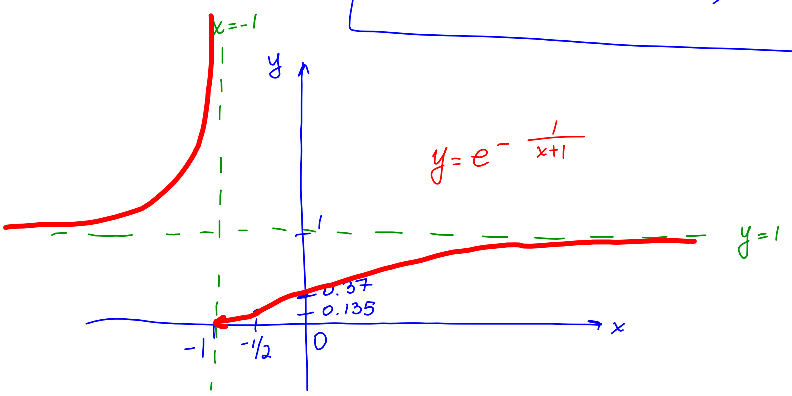
$\boxed{\left(-\tfrac{1}{2}, e^{-2}\right) \approx (-0.5,\ 0.135) \text{ inflection point}}$

$f''(x) > 0$, $f''(x) = \dfrac{e^{-\frac{1}{x+1}}(-2x-1)}{(x+1)^4} > 0$

$\dfrac{e^{-\frac{1}{x+1}}}{(x+1)^4} > 0$ when $x \neq -1$ | $-2x-1 > 0$, $x < -1/2$

f is CU on $(-\infty, -1) \cup (-1, -1/2)$

f is CD on $(-1/2, \infty)$



(b.) $f(x) = \frac{x}{(x-1)^2}$

1. x-, y- intercepts.

$$\frac{x}{(x-1)^2} = 0 \Leftrightarrow x = 0 \qquad \boxed{(0,0)} \text{ point of intersection.}$$

2. asymptotes.

$$\boxed{x = 1 \quad V.A}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x \to 1^-} \frac{x}{(x-1)^2} = \infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x \to 1^+} \frac{x}{(x-1)^2} = \infty$$

$$\lim_{x\to \infty} f(x) = \lim_{x \to \infty} \frac{x}{(x-1)^2} = 0$$

$$\lim_{x\to -\infty} f(x) = 0$$

$$\boxed{y = 0 \quad H.A.}$$

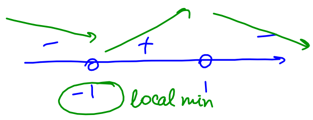
3. Continuity. $x=1$ infinite discontinuity.

continuous on $(-\infty, 1) \cup (1, \infty)$

4. $$f'(x) = \frac{(x-1)^2 - x\,2(x-1)}{(x-1)^4} = \frac{x-1-2x}{(x-1)^3} = \frac{-1-x}{(x-1)^3} = 0$$

$x = -1$ critical number

$f'(x) > 0$ or $\frac{-1-x}{(x-1)^3} > 0$



$$f'(0) = \frac{-1}{(-1)^3} > 0$$

$$f'(2) = \frac{-1-2}{(2-1)^3} < 0$$

$$f'(-2) = \frac{-1+2}{(-1-2)^3} = < 0$$

$\boxed{f \text{ is increasing on } (-1, 1)}$

$\boxed{f \text{ is decreasing on } (-\infty, -1) \cup (1, \infty)}$

$$f(-1) = -\frac{1}{(-1-1)^2} = -\frac{1}{4}$$

$$\boxed{\left(-1, -\tfrac{1}{4}\right) - \text{local min}}$$

5. $f''(x) = \left(- \frac{x+1}{(x-1)^3} \right)'$

$$= - \frac{(x-1)^3 - 3(x-1)^2(x+1)}{(x-1)^6} = - \frac{x-1-3(x+1)}{(x-1)^4} = - \frac{x-1-3x-3}{(x-1)^4}$$

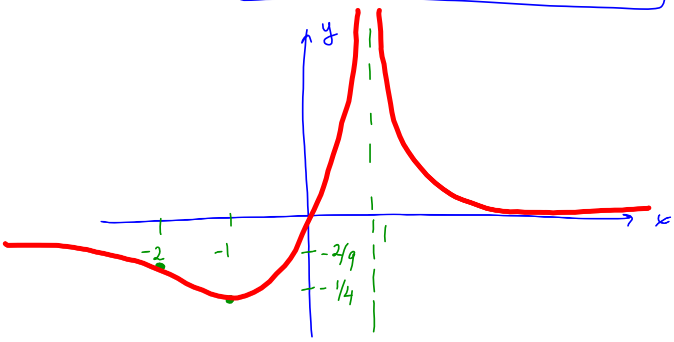
$$= - \frac{-2x-4}{(x-1)^4} = \frac{2x+4}{(x-1)^4} = 0$$

$$x = -2$$

$f''(x) > 0$ when	$x > -2$	f is CU
$f''(x) < 0$ when	$x < -2$	f is CD

$$f(-2) = \frac{-2}{(-2-1)^2} = -\frac{2}{9}$$

$\boxed{(-2, -2/9) \text{ - inflection point}}$



$f(x) = \frac{x}{(x-1)^2}$