

Section 5.5 Applied maximum and minimum problems

Steps in solving applied max and min problems

1. Understand the problem.
2. Draw a diagram.
3. Introduce notation. Assign a symbol to the quantity that is to be minimized or maximized (let us call it Q). Also select symbols (a, b, c, \dots, x, y) for other unknown quantities and label the diagram with these symbols.
4. Express Q in terms of some of the other symbols from step 3.
5. If Q has been expressed as a function of more than one variable in step 4, use the given information to find relationships (in the form of equation) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q . Thus, Q will be given as a function of one variable.
6. Find the absolute maximum or minimum of Q .

Example 1. Find two numbers whose sum is 100 and whose product is a maximum.

$$x + y = 100 \rightarrow y = 100 - x$$

$$P = xy$$

$$P = x(100 - x) = 100x - x^2$$

$$P'(x) = 100 - 2x = 0 \Rightarrow x = 50, y = 100 - 50 = 50$$

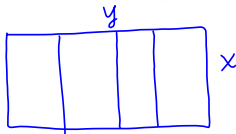
$$0 \leq x \leq 100$$

$$P(0) = 0$$

$$P(50) = 50(100 - 50) = 2500 \text{ absolute max}$$

$$P(100) = 100(100) - 100^2 = 0$$

Example 2. A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of four pens?



$$A = xy$$

$$750 = 2y + 5x$$

$$2y = 750 - 5x$$

$$y = 375 - \frac{5}{2}x$$

$$A = x\left(375 - \frac{5}{2}x\right) = 375x - \frac{5}{2}x^2$$

$$A'(x) = 375 - \frac{5}{2}(2x)$$

$$= 375 - 5x = 0 \Rightarrow x = \frac{375}{5} = 75$$

$$A(75) = \underbrace{375}_{5 \cdot 75}(75) - \frac{5}{2}(75)^2$$

$$= 5(75)^2 - \frac{5}{2}(75)^2 = \frac{5}{2}(75)^2 = \boxed{\frac{28,125}{2}} \begin{matrix} \text{abs} \\ \text{max} \end{matrix}$$

$$0 \leq x \leq 750$$

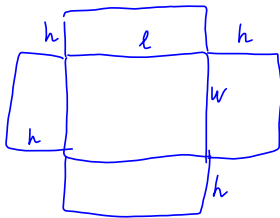
$$A(0) = 0$$

$$A(750) = 375 \cdot 750 - \frac{5}{2}(750)^2$$

$$= \frac{750}{2} \cdot 750 - \frac{5}{2}(750)^2 = -2(750)^2$$

$$\begin{array}{r} 75 \\ 75 \\ \hline 375 \\ 525 \\ \hline 5625 \\ 5 \\ \hline 28125 \end{array}$$

Example 3. A rectangular storage container with an open top is to have a volume of 10 m^3 . The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost for materials for the cheapest such container.



$$L = 2w$$

Total cost:

$$C = lw(10) + 2wh(6) + 2lh(6)$$

$$= 10lw + 12(wh + lh)$$

$$C = 10(2w)w + 12h(w + 2w)$$

$$C = 20w^2 + 36wh$$

$$C(w) = 20w^2 + 36 \cdot w \cdot \frac{5}{w^2}$$

$$C(w) = 20w^2 + \frac{180}{w}$$

$$C'(w) = 40w - \frac{180}{w^2} = 0$$

$$(w^2)4w = \frac{180}{w^2} (w^2)$$

$$4w^3 = 18, \quad w^3 = \frac{18}{4} = \frac{9}{2}, \quad w = \sqrt[3]{\frac{9}{2}}$$

$$C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{9/2}}$$

$$V = lwh = 10$$

$$h = \frac{10}{lw} = \frac{10}{2w^2}$$

$$h = \frac{5}{w^2}$$

Example 4. A particle moves in the xy -plane in such a way that its position at time t is $\vec{r}(t) = \langle 1 + \cos 2t, 2 \sin t \rangle$. Find the particle's maximum speed, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

$$\vec{v}(t) = \vec{r}'(t) = \langle -2 \sin 2t, 2 \cos t \rangle$$

$$s(t) = |\vec{v}(t)| = \sqrt{4 \sin^2 2t + 4 \cos^2 t}$$

$$= 2 \sqrt{4 \sin^2 t \cos^2 t + \cos^2 t}$$

$\sin 2t = 2 \sin t \cos t$

$$= 2 \sqrt{4 \sin^2 t \cos^2 t + \cos^2 t}$$

$$s(t) = 2 \cos t \sqrt{4 \sin^2 t + 1}$$

$$s'(t) = 2(-\sin t) \sqrt{4 \sin^2 t + 1} + 2 \cos t \frac{1}{2} (4 \sin^2 t + 1)^{-1/2} (4 \sin^2 t + 1)'$$

$$= -2 \sin t \sqrt{4 \sin^2 t + 1} + \frac{\cos t}{\sqrt{4 \sin^2 t + 1}} (4(2) \sin t \cos t)$$

$$= \frac{-2 \sin t (4 \sin^2 t + 1) + 8 \sin t \cos^2 t}{\sqrt{4 \sin^2 t + 1}}$$

$$= \frac{-2 \sin t (4 \sin^2 t + 1) + 8 \sin t (1 - \sin^2 t)}{\sqrt{4 \sin^2 t + 1}}$$

$$= \frac{-8 \sin^3 t - 2 \sin t + 8 \sin t - 8 \sin^3 t}{\sqrt{4 \sin^2 t + 1}}$$

$$= \frac{-16 \sin^3 t + 6 \sin t}{\sqrt{4 \sin^2 t + 1}} = 0$$

$$2 \sin t (8 \sin^2 t - 3) = 0$$

$$\sin t = 0 \quad \text{or} \quad \sin^2 t = \frac{3}{8}$$

$$t = 0$$

$$\sin t = \pm \sqrt{\frac{3}{8}}$$

$$t = \sin^{-1} \sqrt{\frac{3}{8}} \approx 0.659$$

$$t = \sin^{-1} \left(-\sqrt{\frac{3}{8}} \right) \approx -0.659$$

$$s(0) = 2 \cos 0 \sqrt{4 \sin^2 0 + 1} = 2$$

$$s(-0.659) = s(0.659) = 2 \cos(0.659) \sqrt{4 \sin^2(0.659) + 1} \approx \boxed{2.5}$$