Section 5.7 **Antiderivatives**

Definition. Function $F(x)$ is called an antiderivative of $f(x)$ on an interval I if

$$\boxed{F'(x) = f(x)}$$

for all $x \in I$.

Theorem 1. If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is a constant.

Table of antidifferentiation formulas

a, C are constants, $F'(x) = f(x)$, $G'(x) = g(x)$

Function	Antiderivative
$af(x)$, a is a constant	$aF(x) + C$
$f(x) \pm g(x)$	$F(x) \pm G(x) + C$
a, a is a constant	$ax + C$
x	$\frac{x^2}{2} + C$
x^n, $n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln\|x\| + C$
e^x	$e^x + C$
a^x	$\frac{a^x}{\ln a} + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} + C$

Example 1. Find the most general antiderivative of the function.

(a.) $f(x) = x^3 - 4x^2 + 17 \cdot 1$

$$F(x) = \frac{x^{3+1}}{3+1} - 4 \cdot \frac{x^{2+1}}{2+1} + 17x + C$$

$$= \boxed{\frac{x^4}{4} + 4\frac{x^3}{3} + 17x + C}$$

(b.) $f(t) = \sin t - \sqrt{t} = \sin t - t^{1/2}$

$$F(t) = -\cos t - \frac{t^{1/2+1}}{1/2+1} + C$$

$$= -\cos t - \frac{t^{3/2}}{3/2} + C$$

$$= \boxed{-\cos t - \frac{2}{3} t^{3/2} + C}$$

(c.) $f(x) = (1 + x^2)\sqrt[3]{x^2} = (1+x^2)x^{2/3} = x^{2/3} + x^2 \cdot x^{2/3} = x^{2/3} + x^{2+2/3} = x^{2/3} + x^{8/3}$

$$F(x) = \frac{x^{2/3+1}}{2/3+1} + \frac{x^{8/3+1}}{8/3+1} + C$$

$$= \frac{x^{5/3}}{5/3} + \frac{x^{11/3}}{11/3} + C$$

$$= \boxed{\frac{3}{5} x^{5/3} + \frac{3}{11} x^{11/3} + C}$$

(d.) $f(x) = \dfrac{x^2 + x + 1}{x} = (x^2+x+1)x^{-1} = x + 1 + \frac{1}{x}$

$$F(x) = \boxed{\frac{x^2}{2} + x + \ln|x| + C}$$

(e.) $f(x) = x^e + \dfrac{5}{1 + x^2} - \dfrac{1}{\sqrt{1 - x^2}}$

$$F(x) = \boxed{\frac{x^{e+1}}{e+1} + 5\tan^{-1}x - \sin^{-1}x + C}$$

Example 2. Find $f(x)$ if

(a.) $f'(x) = 3\sqrt{x} - \frac{1}{\sqrt{x}}$, $f(1) = 2$

$f'(x) = 3x^{1/2} - x^{-1/2}$

$f(x) = 3\frac{x^{1/2+1}}{1/2+1} - \frac{x^{-1/2+1}}{-1/2+1} + C$

$= 3\frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} + C$

$= 3 \cdot \frac{2}{3} x^{3/2} - 2x^{1/2} + C$

$= 2x^{3/2} - 2x^{1/2} + C$

$f(1) = 2 - 2 + C = 2$

$C = 2$

$\boxed{f(x) = 2x^{3/2} - 2x^{1/2} + 2}$

2

(b.) $f''(x) = x$, $f(0) = -3$, $f'(0) = 2$

$f''(x) = x$

$f'(x) = \frac{x^2}{2} + C_1$

$f'(0) = 0 + C_1 = 2$

$C_1 = 2$

$f'(x) = \frac{x^2}{2} + 2$

$f(x) = \frac{x^3}{2 \cdot 3} + 2x + C_2$

$= \frac{x^3}{6} + 2x + C_2$

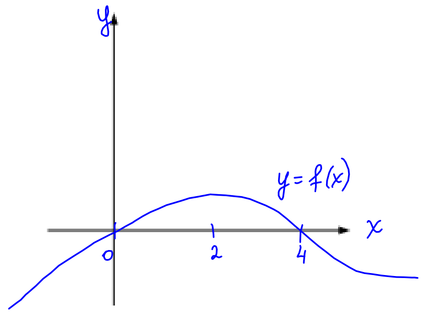
$f(0) = 0 + 0 + C_2 = -3$

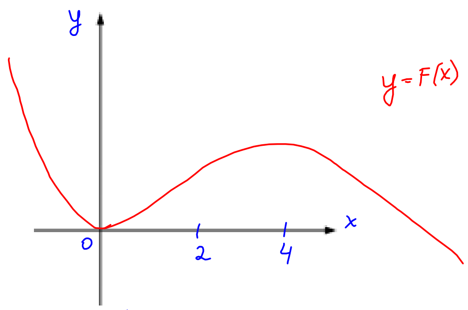
$C_2 = -3$

$\boxed{f(x) = \frac{x^3}{6} + 2x - 3}$

The geometry of antiderivatives

Example 3. Given the graph of a function $f(x)$. Make a rough sketch of of an antiderivative of F, given that $F(0) = 0$.





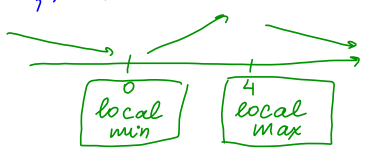
$f(x) > 0$ $(0,4)$

F(x) is increasing on $(0,4)$

$f(x) < 0$ on $(-\infty,0)\cup(4,\infty)$

F(x) is decreasing on $(-\infty,0)\cup(4,\infty)$

$x=0$, $x=4$ – critical values



F is CU

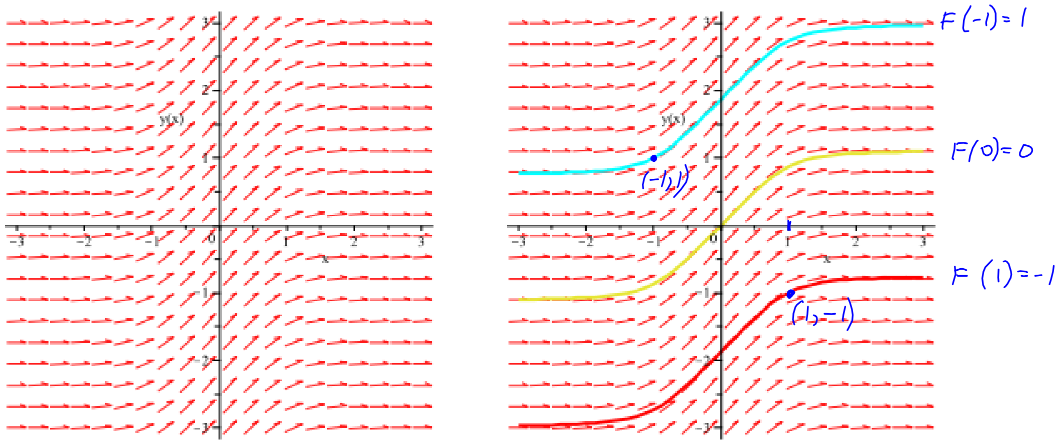
f(x) increases on $(-\infty,2)$

decreases on $(2,\infty)$

F is CD

F has an inflection point @ 2.

Example 4. If $f(x) = 1/(x^4 + 1)$, sketch the graph of those antiderivatives F that satisfy the initial conditions $F(-1) = 1$, $F(0) = 0$, $F(1) = -1$.



Rectilinear motion

If the object has a position function $s = s(t)$, then
$v(t) = s'(t)$ (the position function is an antiderivative for the velocity function),
$a(t) = v'(t)$ (the velocity function is an antiderivative to the acceleration function)

Example 5. A particle is moving with the acceleration $a(t) = 3t + 8$, $s(0) = 1$, $v(0) = -2$. Find the position of the particle.

$a(t) = 3t + 8$

$v(t) = 3\frac{t^2}{2} + 8t + C_1$, $\quad v(0) = 0 + 0 + C_1 = -2 \Rightarrow C_1 = -2$

$V(t) = \frac{3}{2}t^2 + 8t - 2$

$S(t) = \frac{3}{2} \cdot \frac{t^3}{3} + 8 \cdot \frac{t^2}{2} - 2t + C_2$

$= \frac{t^3}{2} + 4t^2 - 2t + C_2$, $\quad S(0) = 0 + 0 - 0 + C_2 = 1 \Rightarrow C_2 = 1$

$\boxed{S(t) = \frac{t^3}{2} + 4t^2 - 2t + 1}$

Antiderivatives of vector functions

Definition. A vector function $\vec{R}(t) =< X(t), Y(t) >$ is called **an antiderivative** of $\vec{r}(t) =< x(t), y(t) >$ on an interval I if $\vec{R}'(t) = \vec{r}(t)$ that is, $X'(t) = x(t)$ and $Y'(t) = y(t)$.

Theorem 2. If $\vec{R}$ is an antiderivative of $\vec{r}$ on an interval I, then the most general antiderivative of $\vec{r}$ on I is

$$\vec{R} + \vec{C}$$

where $\vec{C}$ is an arbitrary constant vector.

Example 6. Find the vector-function that describe the position of particle that has an acceleration $\vec{a}(t) = \cos t\vec{\imath} + e^t\vec{\jmath}$ and $\vec{v}(0) = \vec{\imath} + \vec{\jmath}$, $\vec{r}(0) = \vec{0}$.

$\vec{a}(t) = <\cos t, e^t>$, $\quad \vec{v}(0) = <1, 1>$, $\quad \vec{r}(0) = <0, 0>$

$\vec{v}(t) = <\sin t + C_1, e^t + C_2>$, $\quad \vec{v}(0) = <0 + C_1, e^0 + C_2> = <1, 1>$

$<C_1, 1 + C_2> = <1, 1>$

$C_1 = 1$, $\quad 1 + C_2 = 1 \Rightarrow C_2 = 0$

$\vec{v}(t) = <\sin t + 1, e^t>$

$\vec{r}(t) = <-\cos t + t + C_3, e^t + C_4>$, $\quad \vec{r}(0) = <-1 + 0 + C_3, 1 + C_4> = <0, 0>$

$-1 + C_3 = 0 \Rightarrow C_3 = 1$

$1 + C_4 = 0 \Rightarrow C_4 = -1$

$$\boxed{\vec{r}(t) = <-\cos t + t + 1, e^t - 1>}$$