Section 6.3 **The definite integral**

Definition of a definite integral.

If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ with partition points x_0, x_1,...,x_n, where

$$a = x_0 < x_1 < x_2 < ... < x_n = b$$

Choose points $x_i^* \in [x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. If the limit does exist, then f is called **integrable** on the interval $[a, b]$.

In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and a and b are called the limits of integration; a is the **lower limit** and b is the **upper limit**.

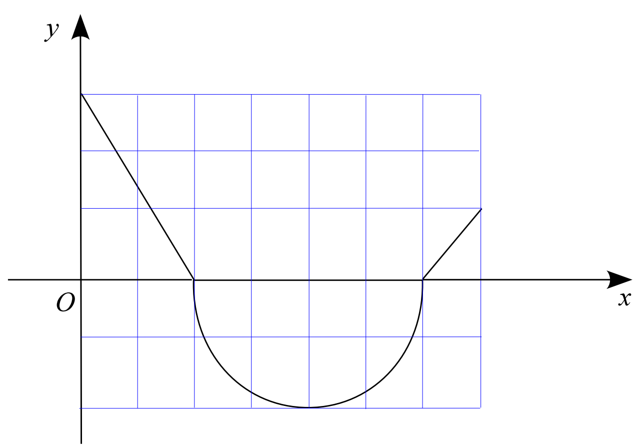
The procedure of calculating an integral is called **integration**.

For the special case where $f(x) \geq 0$, $\int_a^b f(x)dx$ = area under the graph of f from a to b.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

Example 1. Evaluate $\int_0^7 f(x)dx$ if the graph of the function $f(x)$ is



Theorem 1. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

If f has a finite number of discontinuities and these are all jump discontinuities, then f is called **piecewise continuous function**.

Theorem 2. If f is piecewise continuous on $[a, b]$, then f is integrable on $[a, b]$.

f is integrable on $[a, b]$, then f must be **bounded function** on $[a, b]$: that is, there exist a number M such that $|f(x)| \leq M$ for all $x \in [a, b]$.

Let P be a regular partition of $[a, b]$: that is $\Delta x = \Delta x_1 = \Delta x_2 = ... = \Delta x_n = \frac{b-a}{n}$ and $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$,...,$x_n = b$

If we choose x_i^* to be the right endpoint of the ith interval, then $x_i^* = x_i = a + i\Delta x = a + i\frac{b-a}{n}$, so

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

If x_i^* is the midpoint of the interval ith interval, then $x_i^* = \bar{x}_i = (x_{i-1} + x_i)/2$, so

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(\bar{x}_i)$$

Example 2. Evaluate the integral $\int_1^4 (x^2 - 2)dx$

Properties of the definite integral

1. $\int_a^b c\,dx = c(b-a)$, where c is a constant.
2. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is a constant.
3. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
4. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$.
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$.
6. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
7. If $f(x) \geq 0$ for $a < x < b$, then $\int_a^b f(x)dx \geq 0$.
8. If $f(x) \geq g(x)$ for $a < x < b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
9. If $m \leq f(x) \leq M$ for $a < x < b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.
10. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

Example 3. Express the limit as a definite integral

(a.) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$

(b.) $\lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(1+\frac{2i}{n}\right)^5 - 6\right]\frac{2}{n}$

Example 4. Write the given sum or difference as a single integral

(a.) $\int_1^3 f(x)dx + \int_3^6 f(x)dx + \int_6^1 2f(x)dx$

(b.) $\int_{2}^{10} f(x)dx - \int_{2}^{7} f(x)dx$