Chapter 6. **Integrals**
Section 6.1 **Sigma notation**

Definition. If a_m, a_{m+1}, a_{m+2},...,a_n are real numbers and m and n are integers such that $m \leq n$, then

$$a_m + a_{m+1} + a_{m+2} + ... + a_n = \sum_{i=m}^{n} a_i \quad , \qquad a_i = f(i)$$

Example 1. Write the sum in sigma notation.

1. 1+2+3+...+10 $= \boxed{\sum_{i=1}^{10} i}$

   $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, ..., $a_{10} = 10$

   $a_i = i$

2. 1+2+4+8+16+32 $= \boxed{\sum_{i=1}^{6} 2^{i-1}} = \boxed{\sum_{i=0}^{5} 2^{i}}$

   $a_1 = 1 = 2^0$, $a_2 = 2 = 2^1$, $a_3 = 4 = 2^2$, $a_4 = 8 = 2^3$, $a_5 = 16 = 2^4$, $a_6 = 32 = 2^5$

   $a_i = 2^{i-1}$

3. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \boxed{\sum_{i=1}^{5} \frac{1}{i^2}}$

   $a_1 = 1$, $a_2 = \frac{1}{4} = \frac{1}{2^2}$, $a_3 = \frac{1}{9} = \frac{1}{3^2}$, $a_4 = \frac{1}{16} = \frac{1}{4^2}$, $a_5 = \frac{1}{25} = \frac{1}{5^2}$

   $a_i = \frac{1}{i^2}$, $i = 1,..,5$

4. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + ... + \frac{23}{27} = \boxed{\sum_{i=3}^{23} \frac{i}{i+4}}$

   $a_3 = \frac{3}{7} = \frac{3}{3+4}$, $a_4 = \frac{4}{8} = \frac{4}{4+4}$, $a_5 = \frac{5}{5+4}$, ..., $a_{23} = \frac{23}{23+4}$

   $a_i = \frac{i}{i+4}$, $i = 3,...,23$

5. $1 + 3 + 5 + 7 + 9 + 11 + ... + 35 = \boxed{\sum_{i=0}^{17} 2i+1}$

   $a_0 = 1 = 2(0)+1$, $a_1 = 3 = 2(1)+1$, $a_2 = 5 = 2(2)+1$, ..., $a_{17} = 2(17)+1$

   $a_i = 2i+1$, $i = 0,...,17$

Example 2. Write the sum in expanded form

1. $\sum_{i=4}^{9} \sqrt{i}$ $= \boxed{\sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8} + \sqrt{9}}$

2. $\sum_{i=0}^{4} 3^i$ $= 3^0 + 3^1 + 3^2 + 3^3 + 3^4$
$= \boxed{1 + 3 + 9 + 27 + 81}$

3. $\sum_{i=1}^{6} \frac{1}{i+1}$ $= \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{4+1} + \frac{1}{5+1} + \frac{1}{6+1}$
$= \boxed{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}$

4. $\sum_{i=1}^{n} 2i$, here n is an integer
$= 2 + 2(2) + 2(3) + 2(4) + \ldots + 2n$
$= \boxed{2 + 4 + 6 + 8 + \ldots + 2n}$

Theorem. If c is any constant (this means that c does not depend on i), then

1. $\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$
2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
3. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

- $\boxed{\sum_{i=1}^{n} 1 = n}$
- $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$
- $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$
- $\boxed{\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2}$

- $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
- $\boxed{\sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n-1)}{r-1}}$

Example 3. Evaluate

1. $\sum_{i=1}^{n}(3+2i)^2 = \sum_{i=1}^{n}(9+12i+4i^2) = \sum_{i=1}^{n}9+\sum_{i=1}^{n}12i+\sum_{i=1}^{n}4i^2$

$$= 9\sum_{i=1}^{n}1+12\sum_{i=1}^{n}i+4\sum_{i=1}^{n}i^2$$

$$= \boxed{9n+12\frac{n(n+1)}{2}+4\frac{n(n+1)(2n+1)}{6}}$$

2. $\sum_{i=1}^{n}i(i+1)(i+2) = \sum_{i=1}^{n}(i^2+i)(i+2) = \sum_{i=1}^{n}(i^3+2i^2+i^2+2i)$

$$= \sum_{i=1}^{n}(i^3+3i^2+2i) = \sum_{i=1}^{n}i^3+3\sum_{i=1}^{n}i^2+2\sum_{i=1}^{n}i$$

$$= \boxed{\left[\frac{n(n+1)}{2}\right]^2+3\cdot\frac{n(n+1)(2n+1)}{6}+2\frac{n(n+1)}{2}}$$

$$\boxed{\sum_{i=1}^{n}ar^{i-1}=\frac{a(r^n-1)}{r-1}}$$

3. $\sum_{i=1}^{n}(2i+2^i)$

$$= 2\sum_{i=1}^{n}i+\sum_{i=1}^{n}2^i = 2\sum_{i=1}^{n}i+\sum_{i=1}^{n}\underbrace{2}_{a}\cdot\underbrace{2}_{r}{}^{i-1} = 2\frac{n(n+1)}{2}+\frac{2(2^n-1)}{2-1}$$

$$= \boxed{n(n+1)+2(2^n-1)}$$

Example 4. Find $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i}{n}\right)^2$ $= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\,\frac{i^2}{n^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2}{n^3}$

$$= \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty}\frac{1}{n^{\cancel{3}2}}\,\frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty}\frac{\cancel{n}(1+\cancel{\tfrac{1}{n}}^{\,0})\,\cancel{n}(2+\cancel{\tfrac{1}{n}}^{\,0})}{6\cancel{n^2}} = \frac{2}{6} = \boxed{\frac{1}{3}}$$