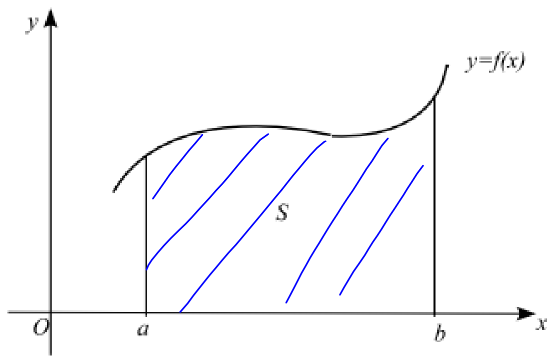
Section 6.2 **Area**

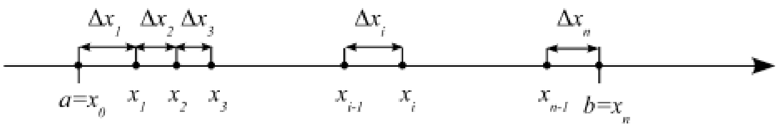
Problem. Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

$$S = \{(x, y) : \ a \le x \le b, \ 0 \le y \le \ f(x)\}$$



We start by subdividing the interval $[a, b]$ into smaller subintervals by choosing partition points x_0, x_1, x_2,...,x_n so that

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$



Then the n subintervals are

$$[x_0, x_1], [x_1, x_2]...[x_{n-1}, x_n]$$

This subdivision is called a **partition** of $[a, b]$ and we denote it by P. We use the notation Δx_i for the length of the ith subinterval $[x_{i-1}, x_i]$.

$$\Delta x_i = x_i - x_{i-1}$$

The length of the longest subinterval is denoted by $\|P\|$ and is called the **norm** of P.

$$\|P\| = \max\{\Delta x_1, \Delta x_2, ..., \Delta x_n\}$$

Partition $[a,b]$ into n subintervals of equal length Δx, then

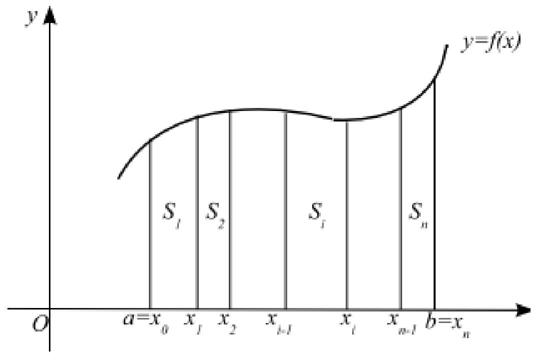
- $\boxed{\Delta x = \frac{b-a}{n}}$

- Partition points:

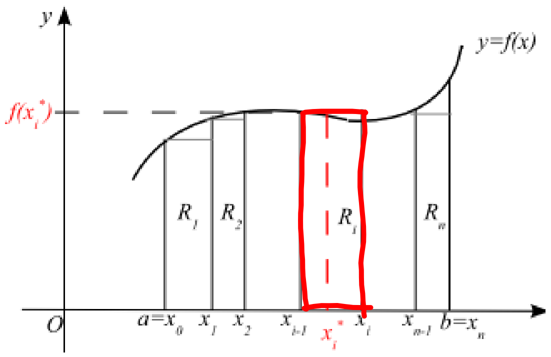
$$x_0 = a$$
$$x_1 = a + \Delta x$$
$$x_2 = x_1 + \Delta x = (a + \Delta x) + \Delta x = a + 2\Delta x$$
$$x_3 = x_2 + \Delta x = a + 3\Delta x$$
$$\cdots$$
$$x_i = a + i\Delta x$$
$$\cdots$$
$$x_n = b$$

- $\|P\| \to 0$ is equivalent to $n \to \infty$

By drawing the lines $x = a$, $x = x_1$, ... $x = b$, we use the partition P to divide the region S into strips S_1, S_2,...,S_n.



We choose a number x_i^* in each subinterval $[x_{i-1}, x_i]$ and construct a rectangle R_i with base Δx_i and height $f(x_i^*)$.



$x_{i-1} \le x_i^* \le x_i$

The area of the ith rectangle R_i is

$$\boxed{A_i = f(x_i^*)\Delta x_i}$$

The n rectangles R_1, R_2,...,R_n form a polygonal approximation to the region S.

$$\boxed{A(S) \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*)\Delta x_i}$$

This approximation becomes better and better as the strips become thinner and thinner, that is, as $\|P\| \to 0$. Then

$$\boxed{A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i}$$

Example 1. Determine a region whose area is equal to

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt{1+\frac{3i}{n}} = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

DO NOT EVALUATE THE LIMIT.

$f(x_i^*) = \sqrt{1+\frac{3i}{n}}$, $\Delta x_i = \frac{3}{n}$ ← does not depend on i.

subintervals of equal length.

$\Delta x = \frac{3}{n} = \frac{b-a}{n}$

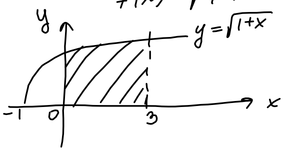
$[0,3]$

$x_0 = 0,\quad x_i = x_0 + i\Delta x$

$= i\cdot\frac{3}{n}$

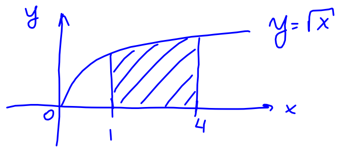
$x_i^* = x_i = \frac{3i}{n}$

$f(x) = \sqrt{1+x}$

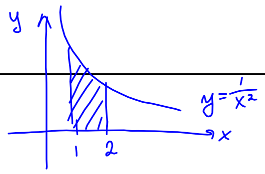


$x_i^* = \underbrace{1}_{a} + \frac{3i}{n} \Rightarrow [1,4]$

$f(x) = \sqrt{x}$



Example 2. Find the area under the curve $y = 1/x^2$ from 1 to 2. Use four subintervals of equal length and take x_i^* to be the midpoint of the ith subinterval.



- length of subintervals $\Delta x = \frac{2-1}{4} = \frac{1}{4}$

- Partition points:

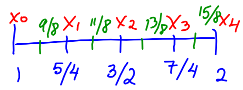
$x_0 = 1$

$x_1 = x_0 + \Delta x = 1 + \frac{1}{4} = \frac{5}{4}$

$x_2 = x_1 + \Delta x = 1 + \frac{5}{4}\ \ = \frac{3}{2}$

$x_3 = x_2 + \Delta x = 1 + \frac{3}{2} = \frac{7}{4}$

$x_4 = 2$



- Find x_i^*.

x_1^* is the midpoint for $[1, \frac{5}{4}]$

$x_1^* = \frac{1+\frac{5}{4}}{2} = \frac{9}{8}$, $f(x_1^*) = f(\frac{9}{8}) = \frac{1}{(9/8)^2} = \frac{64}{81}$

x_2^* is the midpoint for $[\frac{5}{4}, \frac{3}{2}]$

$x_2^* = \frac{\frac{5}{4}+\frac{3}{2}}{2} = \frac{11}{8}$, $f(x_2^*) = f(\frac{11}{8}) = \frac{1}{(11/8)^2} = \frac{64}{121}$

x_3^* is the midpoint for $[\frac{3}{2}, \frac{7}{4}]$

$x_3^* = \frac{\frac{3}{2}+\frac{7}{4}}{2} = \frac{13}{8}$, $f(x_3^*) = f(\frac{13}{8}) = \frac{1}{(13/8)^2} = \frac{64}{169}$

x_4^* is the midpoint for $[\frac{7}{4}, 2]$

$x_4^* = \frac{\frac{7}{4}+2}{2} = \frac{15}{8}$, $f(x_4^*) = f(\frac{15}{8}) = \frac{1}{(15/8)^2} = \frac{64}{225}$

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x$$

$$= \underbrace{\frac{1}{4}}_{\Delta x}\left(\frac{64}{81} + \frac{64}{121} + \frac{64}{169} + \frac{64}{225}\right) = \boxed{16\left(\frac{1}{81} + \frac{1}{121} + \frac{1}{169} + \frac{1}{225}\right)}$$

≈ 0.4955

Example 3. Find the area under the curve $y = x^2 + 3x - 2$ from 1 to 4. Use equal subintervals ~~and take x_i^* to be the~~ right endpoint of the ith subinterval.

- Length of subintervals $\Delta x = \frac{4-1}{n} = \frac{3}{n}$
- Partition points: $x_0 = 1$

  $x_1 = 1 + \Delta x = 1 + \frac{3}{n}$

  $x_2 = 1 + 2\Delta x = 1 + 2\cdot\frac{3}{n}$

  $\dots$

  $x_i = 1 + i\cdot\frac{3}{n}$

  $\dots$

  $x_n = 4$

- $x_i^* = x_i = 1 + \frac{3i}{n}$, $[x_{i-1}, x_i]$

$$f(x_i^*) = (x_i^*)^2 + 3x_i^* - 2 = \left(1 + \frac{3i}{n}\right)^2 + 3\left(1 + \frac{3i}{n}\right) - 2$$

$$= 1 + \frac{6i}{n} + \frac{9i^2}{n^2} + 3 + \frac{9i}{n} - 2$$

$$= \frac{9i^2}{n^2} + \frac{15i}{n} + 2$$

- $$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} + \frac{15i}{n} + 2\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\sum_{i=1}^{n} \frac{9i^2}{n^2} + \sum_{i=1}^{n} \frac{15i}{n} + \sum_{i=1}^{n} 2\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{9}{n^2} \underbrace{\sum_{i=1}^{n} i^2}_{\frac{n(n+1)(2n+1)}{6}} + \frac{15}{n} \underbrace{\sum_{i=1}^{n} i}_{\frac{n(n+1)}{2}} + 2 \underbrace{\sum_{i=1}^{n} 1}_{n}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{9}{\cancel{n^2}} \frac{\cancel{n}(n+1)(2n+1)}{6} + \frac{15}{\cancel{n}} \frac{\cancel{n}(n+1)}{2} + 2n\right]$$

$$= \lim_{n\to\infty} \left(\frac{9(n+1)(2n+1)}{\cancel{2n^2}}^{\frac{9(2)}{2}} + \frac{45(n+1)}{\cancel{2n}}^{\frac{45}{2}} + \frac{6\cancel{n}}{\cancel{n}}\right) = 9 + \frac{45}{2} + 6 = \boxed{37.5}$$