Section 6.3 **The definite integral**

Definition of a definite integral.

If f is a function defined on a closed interval $[a,b]$, let P be a partition of $[a,b]$ with partition points x_0, x_1,...,x_n, where

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

Choose points $x_i^* \in [x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Riemann sum is $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$

if this limit exists. If the limit does exist, then f is called **integrable** on the interval $[a,b]$.

In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and a and b are called the limits of integration; a is the **lower limit** and b is the **upper limit**.

The procedure of calculating an integral is called **integration**.

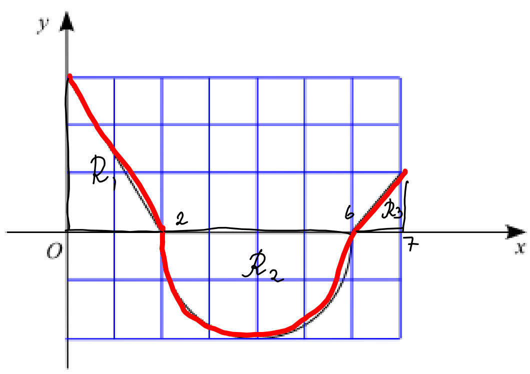
b

For the special case where $f(x) \geq 0$, $\int_a^b f(x)dx =$ area under the graph of f from a to b.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

Example 1. Evaluate $\int_0^7 f(x)dx$ if the graph of the function $f(x)$ is



$$\int_0^7 f(x)dx = A(R_1) - A(R_2) + A(R_3)$$

$$= \frac{1}{2}(2)(3) - \frac{\pi}{2}(2^2) + \frac{1}{2}(1)(1)$$

$$= 3 + \frac{1}{2} - 2\pi$$

$$= \boxed{\frac{7}{2} - 2\pi}$$

Theorem 1. If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

If f has a finite number of discontinuities and these are all jump discontinuities, then f is called **piecewise continuous function**.

Theorem 2. If f is piecewise continuous on $[a,b]$, then f is integrable on $[a,b]$.

f is integrable on $[a,b]$, then f must be **bounded function** on $[a,b]$: that is, there exist a number M such that $|f(x)| \le M$ for all $x \in [a,b]$.

Let P be a regular partition of $[a,b]$: that is $\Delta x = \Delta x_1 = \Delta x_2 = ... = \Delta x_n = \frac{b-a}{n}$ and $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$,...,$x_n = b$

If we choose x_i^* to be the right endpoint of the ith interval, then $x_i^* = x_i = a + i\Delta x = a + i\frac{b-a}{n}$, so

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

If x_i^* is the midpoint of the [redacted] ith interval, then $x_i^* = \bar{x}_i = (x_{i-1} + x_i)/2$, so

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(\bar{x}_i)$$

Example 2. Evaluate the integral $\int_1^4 (x^2 - 2)dx$

- Partition $[1,4]$ into n subintervals of equal length

$\Delta x = \frac{4-1}{n} = \frac{3}{n}$

- Partition points:

$x_0 = 1$

$x_1 = 1 + \Delta x = 1 + \frac{3}{n}$

$x_2 = 1 + 2\Delta x = 1 + 2\cdot\frac{3}{n}$

$x_i = 1 + i\Delta x = 1 + i\cdot\frac{3}{n}$

$x_n = 4$

- $x_i^* = x_i = 1 + i\frac{3}{n}$

$f(x_i^*) = \left(1 + \frac{3i}{n}\right)^2 - 2$

$= 1 + \frac{6i}{n} + \frac{9i^2}{n^2} - 2$

$= \frac{9i^2}{n^2} + \frac{6i}{n} - 1$

$$\int_1^4 (x^2-2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{9i^2}{n^2} + \frac{6i}{n} - 1\right)}_{f(x_i^*)} \underbrace{\frac{3}{n}}_{f(x)}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left(\sum_{i=1}^{n} \frac{9i^2}{n^2} + \sum_{i=1}^{n} \frac{6i}{n} - \sum_{i=1}^{n} 1\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\left(\frac{9}{n^2} \underbrace{\sum_{i=1}^{n} i^2}_{\frac{n(n+1)(2n+1)}{6}} + \frac{6}{n} \underbrace{\sum_{i=1}^{n} i}_{\frac{n(n+1)}{2}} - \underbrace{\sum_{i=1}^{n} 1}_{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\left(\frac{9}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{6}{n} \cdot \frac{n(n+1)}{2} - n\right)$$

$$= \lim_{n\to\infty} \left(\frac{9(n+1)(2n+1)}{2n^2} + \frac{9(n+1)}{n} - \frac{3n}{n}\right) = 9 + 9 - 3 = \boxed{15}$$

$$\int_1^4 (x^2-2)\,dx = \left[\frac{x^3}{3} - 2x\right]_1^4 = \frac{4^3}{3} - 2(4) - \left(\frac{1}{3} - 2\right)$$

$$= \frac{64}{3} - 8 - \frac{1}{3} + 2 = \frac{63}{3} - 6 = 21 - 6 = \boxed{15}$$

Properties of the definite integral

1. $\int_a^b c\,dx = c(b-a)$, where c is a constant.

2. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is a constant.

3. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

4. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$.

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$.

6. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

7. If $f(x) \geq 0$ for $a < x < b$, then $\int_a^b f(x)dx \geq 0$.

8. If $f(x) \geq g(x)$ for $a < x < b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

9. If $m \leq f(x) \leq M$ for $a < x < b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

10. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

Example 3. Express the limit as a definite integral

(a.) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{i^4}{n^4}}_{f(x_i^*)} \cdot \underbrace{\frac{1}{n}}_{\Delta x} = \boxed{\int_0^1 x^4 dx}$

$f(x_i^*) = \frac{i^4}{n^4}$, if $x_i^* = \frac{i}{n}$, then $f(x) = x^4$

$\frac{b-a}{n} = \frac{1}{n} \Rightarrow b-a=1$, $a=0$

$x_i^* = a + i\Delta x$

$\Delta x = \frac{1}{n}$

$x_i^* = \frac{i}{n} \Rightarrow a=0$

(b.) $\lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(1+\frac{2i}{n}\right)^5 - 6\right] \frac{2}{n}$

$\Delta x = \frac{2}{n} = \frac{b-a}{n} \Rightarrow b-a=2$

$f(x_i^*) = 3\left(1+\frac{2i}{n}\right)^5 - 6 \Rightarrow f(x) = 3x^5 - 6$

$x_i^* = 1 + \frac{2i}{n}$

$= a + i \cdot \Delta x \Rightarrow a = 1$

$b = a+2 = 3$

$= \boxed{\int_1^3 (3x^5 - 6)\, dx}$

Example 4. Write the given sum or difference as a single integral

(a.) $\int_1^3 f(x)dx + \int_3^6 f(x)dx + \int_6^1 2f(x)dx$

$\int_1^6 f(x)dx - \int_1^6 2f(x)\,dx = \int_1^6 \left(f(x) - 2f(x)\right)dx = \boxed{-\int_1^6 f(x)\,dx}$

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(b.) $\int_2^{10} f(x)dx - \int_2^7 f(x)dx$

$\int_2^7 f(x)\,dx + \int_7^{10} f(x)dx - \int_2^7 f(x)dx = \boxed{\int_7^{10} f(x)\,dx}$