Section 6.4 **The fundamental theorem of calculus.**

Theorem. Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$, where F is an antiderivative of f.

Example 1. Find the derivative of the function.

1. $g(x) = \int_{\pi}^{x} \frac{1}{1+t^4} dt$

$g'(x) = f(x)$, where $f(x) = \frac{1}{1+x^4}$

$\boxed{g'(x) = \frac{1}{1+x^4}}$

2. $f(x) = \int_x^4 (2+\sqrt{t})^8 dt = -\int_4^x (2+\sqrt{t})^8 dt$

$\boxed{f'(x) = -(2+\sqrt{x})^8}$

3. $y = \int_{\tan x}^{17} \sin(t^4) dt = -\int_{17}^{\tan x} \sin(t^4) dt$

$u = \tan x$

$y' = -\sin(u^4) \cdot \frac{du}{dt}$

$= \boxed{-\sin(\tan^4 x) \cdot \sec^2 x}$

Example 2. Evaluate the integral.

1. $\displaystyle\int_2^6 \frac{1+\sqrt{y}}{y^2}\,dy = \int_2^6 (1+\sqrt{y})\,y^{-2}\,dy = \int_2^6 \big(y^{-2} + \overbrace{y^{1/2}\cdot y^{-2}}^{y^{1/2-2}}\big)\,dy$

$$= \int_2^6 \left(y^{-2} + y^{-3/2}\right) dy = \left[\frac{y^{-2+1}}{-2+1} + \frac{y^{-3/2+1}}{-3/2+1}\right]_2^6 = \left[-\frac{1}{y} - \frac{2}{\sqrt{y}}\right]_2^6$$

1

$$= \left[-\frac{1}{6} - \frac{2}{\sqrt{6}}\right] - \left[-\frac{1}{2} - \frac{2}{\sqrt{2}}\right] = -\frac{1}{6} - \frac{2}{\sqrt{6}} + \frac{1}{2} + \frac{2}{\sqrt{2}}$$

$$= \boxed{\frac{1}{3} - \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{2}}}$$

---

2. $\displaystyle\int_0^2 f(x)\,dx$, where $f(x) = \begin{cases} x^4, & 0 \le x < 1 \\ x^5, & 1 \le x \le 2 \end{cases}$

$$= \int_0^1 x^4\,dx + \int_1^2 x^5\,dx = \frac{x^{4+1}}{4+1}\Big]_0^1 + \frac{x^{5+1}}{5+1}\Big]_1^2$$

$$= \frac{x^5}{5}\Big]_0^1 + \frac{x^6}{6}\Big]_1^2 = \frac{1}{5} - 0 + \frac{2^6}{6} - \frac{1}{6}$$

$$= \frac{64}{6} - \frac{1}{6} + \frac{1}{5} = 21 + \frac{1}{5} = \boxed{\frac{106}{5}}$$

Example 3. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 2t - 8$.

1. Find the displacement of the particle during the time period $1 \le t \le 6$.

$t=6$ $\quad$ $t=1$

~~displacement~~ displacement.

$$D = \int_1^6 v(t)\,dt = \int_1^6 (t^2 - 2t - 8)\,dt = \left[\frac{t^3}{3} - \frac{\cancel{2}t^2}{\cancel{2}} - 8t\right]_1^6$$

$$= \frac{6^3}{3} - 6^2 - 8(6) - \left(\frac{1}{3} - 1 - 8\right)$$

$$= 72 - 36 - 48 - \frac{1}{3} + 9 = -3 - \frac{1}{3} = -\frac{10}{3}$$

$$\text{displacement} = \left|-\frac{10}{3}\right| = \boxed{\frac{10}{3}}$$

2. Find the distance traveled during this time period.

Find where $v(t) > 0$

$t^2 - 2t - 8 > 0$

$(t-4)(t+2) > 0$

$-$ $\quad$ $+$

$-2 \quad 1 \quad 4 \quad 6$

$v(2) = (2-4)(2+2) < 0$

$v(5) = (5-4)(5+2) > 0$

$v(t) < 0$ on $(1,4)$

$v(t) > 0$ on $(4,6)$

$$\int_1^6 |v(t)|\,dt = \int_1^4 (-v(t))\,dt + \int_4^6 v(t)\,dt$$

$$= \int_1^4 (-t^2 + 2t + 8)\,dt + \int_4^6 (t^2 - 2t - 8)\,dt$$

$$= \left[-\frac{t^3}{3} + \frac{2t^2}{2} + 8t\right]_1^4 + \left[\frac{t^3}{3} - \frac{2t^2}{2} - 8t\right]_4^6$$

$$= -\frac{4^3}{3} + 4^2 + 8(4) - \left(-\frac{1}{3} + 1 + 8\right) + \frac{6^3}{3} - 6^2 - 8(6) - \left(\frac{4^3}{3} - 4^2 - 8(4)\right)$$

$$= -\frac{64}{3} + 16 + 32 + \frac{1}{3} - 9 + 72 - 36 - 48 - \frac{64}{3} + 16 + 32 = \boxed{\frac{122}{3}}$$