Derivatives of vector functions

Definition. The **derivative of a vector function** $\mathbf{r}(t)$ **at a number** a, denoted by $\mathbf{r}\,'(a)$, is

$$\boxed{\mathbf{r}\,'(a) = \lim_{h\to 0}\frac{\mathbf{r}(a+h)-\mathbf{r}(a)}{h} = \lim_{t\to a}\frac{\mathbf{r}(t)-\mathbf{r}(a)}{t-a}}$$

if the limits exist.

If $\mathbf{r}(t) =< x(t), y(t) >$ is a vector function, then

$$\mathbf{r}\,'(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} = \left\langle \lim_{h\to 0}\frac{x(t+h)-x(t)}{h}, \lim_{h\to 0}\frac{y(t+h)-y(t)}{h}\right\rangle =< x'(t), y'(t) >$$

if both $x'(t)$ and $y'(t)$ exist. Thus,

$$\boxed{\mathbf{r}'(t) =< x'(t), y'(t) >}$$

Example 1. Find the domain and the derivative of the vector function $\mathbf{r}(t) =< t^2-4, \sqrt{t-4} >$.

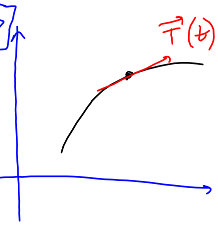
Domain: $t-4 \geq 0$ or $\boxed{t \geq 4}$

$\vec{r}\,'(t) = < (t^2-4)', (\sqrt{t-4})' >$

$= \boxed{< 2t, \frac{1}{2}(t-4)^{-1/2} >}$

Example 2. Find a tangent vector of unit length for the curve given by $\mathbf{r}(t) = 2\sin t\mathbf{i} + 3\cos t\mathbf{j}$ at the point where $t = \pi/6$.

$\boxed{\text{tangent vector } \vec{T}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t)\rangle}$



$\vec{r}(t) = \langle 2\sin t, 3\cos t\rangle$

tangent vector $\vec{T}(t) = \vec{r}\,'(t) = \langle (2\sin t)', (3\cos t)'\rangle$

$\vec{T}(t) = \langle 2\cos t, -3\sin t\rangle$

$\vec{T}\left(\frac{\pi}{6}\right) = \langle 2\cos\frac{\pi}{6}, -3\sin\frac{\pi}{6}\rangle$

$= \langle 2\cdot\frac{\sqrt{3}}{2}, -3\cdot\frac{1}{2}\rangle = \langle \sqrt{3}, -\frac{3}{2}\rangle$

$\left|\vec{T}\left(\frac{\pi}{6}\right)\right| = \sqrt{3+\frac{9}{4}} = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$

unit tangent vector $= \dfrac{1}{\frac{\sqrt{21}}{2}}\langle \sqrt{3}, -\frac{3}{2}\rangle = \boxed{\frac{2}{\sqrt{21}}\langle \sqrt{3}, -\frac{3}{2}\rangle}$

Definition. If $\mathbf{r}(t) =< x(t), y(t) >$ is a vector function representing the position of a particle at time t, then

velocity at time t is

$$\boxed{\mathbf{v}(t) = \mathbf{r}'(t) =< x'(t), y'(t) >}$$

speed at time t is

$$\boxed{s = |\mathbf{v}(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}}$$

acceleration $\boxed{\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)}$

Example 3. The vector function $\mathbf{r}(t) =< t, 25t - 5t^2 >$ represents the position of a particle at time t. Find the velocity, ~~and the~~ speed and acceleration at $t = 1$.

Velocity $\quad \vec{v}(t) = \vec{r}'(t) = < t', (25t - 5t^2)' >$

$$= < 1, 25 - 10t >$$

$$\vec{v}(1) = \vec{r}'(1) = < 1, 25 - 10 >$$

$$= < 1, 15 >$$

speed @ $t = 1$ $\quad s(1) = |\vec{r}'(1)| = \sqrt{1^2 + 15^2} = \sqrt{226}$

acceleration $\quad \vec{a}(t) = \vec{r}''(t) = \vec{v}'(t) = < (1)', (25 - 10t)' >$

$$\vec{a}(t) = < 0, -10 >$$