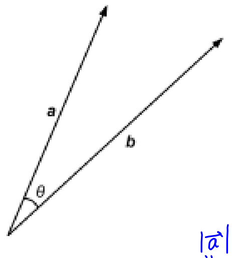
The dot product.

Definition. The **dot** or **scalar product** of two nonzero vectors $\vec{a}$ and $\vec{b}$ is the number

$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$. If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, we define $\vec{a}\cdot\vec{b} = 0$.

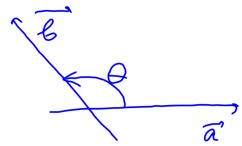


Example 1. If the vectors $\vec{a}$ and $\vec{b}$ have lengths $\overset{|\vec{a}|}{2}$ and $\overset{|\vec{b}|}{6}$, and the angle between them is $\underbrace{\pi/4}_{\theta}$, find $\vec{a}\cdot\vec{b}$.

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta = 2\cdot 6\cos\frac{\pi}{4} = 12\cdot\frac{\sqrt{2}}{2} = \boxed{6\sqrt{2}}$$

If $\vec{a} =< a_1, a_2 >$ and $\vec{b} =< b_1, b_2 >$, then

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2$$



$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

if $0<\theta<\frac{\pi}{2} \Rightarrow \vec{a}\cdot\vec{b}>0$

if $\frac{\pi}{2}<\theta<\pi \Rightarrow \vec{a}\cdot\vec{b}<0$

Example 2. Find $\vec{a}\cdot\vec{b}$ if $\vec{a} =< 2, 3 >$ and $\vec{b} = \vec{\imath} - 3\vec{\jmath}$ $= <1, -3>$

$$\vec{a}\cdot\vec{b} = 2(1) + 3(-3) = 2 - 9 = -7, \quad \frac{\pi}{2}<\theta<\pi$$

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called perpendicular or orthogonal if the angle between them is $\pi/2$.

Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b} = 0$.

Example 3. Find the angle between the vectors $\vec{a} = 6\vec{\imath} - 2\vec{\jmath}$ and $\vec{b} =< 1, 1 >$.

$\vec{a} = <6, -2>$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta \Rightarrow \boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}}$$

$\vec{a}\cdot\vec{b} = 6(1) - 2(1) = 4$

$|\vec{a}| = \sqrt{6^2 + (-2)^2} = \sqrt{40} = 2\sqrt{10}$

$|\vec{b}| = \sqrt{1+1} = \sqrt{2}$

$$\cos\theta = \frac{4}{2\sqrt{10}\cdot\sqrt{2}} = \frac{2}{\sqrt{20}} = \frac{2}{2\sqrt{5}} = \boxed{\frac{1}{\sqrt{5}}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)} \approx 63^\circ$$

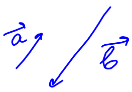
Example 4. Determine whether the given vectors are orthogonal, parallel, or neither.

$\vec{a} \perp \vec{b}$ if and only if $\vec{a} \cdot \vec{b}$

$\vec{a} \parallel \vec{b}$ if and only if $\vec{a} = c \cdot \vec{b}$, c is a constant

(a.) $\vec{a} =< 1, -2 >$, $\vec{b} = -2\vec{\imath} + 4\vec{\jmath}$ $= <-2, 4>$

$\vec{b} = -2\vec{a}$



$\boxed{\text{parallel}}$

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$

$\vec{a} \parallel \vec{b}$ $\quad \theta = 0$ or $\theta = \pi$

$\cos\theta = 1$ or $\cos\theta = -1$

(b.) $\vec{a} =< 3, 1 >$, $\vec{b} =< -3, 9 >$

$\vec{a} \cdot \vec{b} = 3(-3) + 1(9) = 0$

$\boxed{\text{perpendicular}}$

(c.) $\vec{a} = -\vec{\imath} + 4\vec{\jmath}$, $\vec{b} = 3\vec{\imath} - 2\vec{\jmath}$

$\vec{a} = <-1, 4>$, $\vec{b} = <3, -2>$

$\vec{a} \cdot \vec{b} = <-1, 4> \cdot <3, -2> = -1(3) + 4(-2) = -11 \neq 0$ $\quad \boxed{\text{not perpendicular}}$

$|\vec{a}| = \sqrt{1+16} = \sqrt{17}$, $\quad |\vec{b}| = \sqrt{9+4} = \sqrt{13}$

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{-11}{\sqrt{17}\sqrt{13}} \neq \pm 1$ $\quad \boxed{\text{not parallel}}$

Properties of the dot product If $\vec{a}$, $\vec{b}$, an $\vec{c}$ are vectors and k is a scalar, then

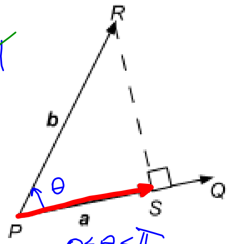
1. $\vec{a} \cdot \vec{a} = |a|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a}(k\vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

Vector and scalar projections.

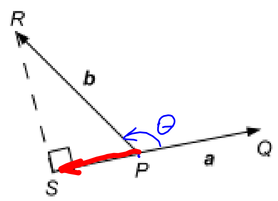
$\text{comp}_{\vec a}\vec b = |\vec{PS}| = |\vec b|\cos\theta$

$= \cancel{|\vec b|}\cdot \dfrac{\vec a\cdot\vec b}{|\vec a|\cdot\cancel{|\vec b|}}$

$\boxed{\text{comp}_{\vec a}\vec b = \dfrac{\vec a\cdot \vec b}{|\vec a|}}$



$0<\theta<\frac{\pi}{2}$



$\frac{\pi}{2}<\theta<\pi$

$\text{comp}_{\vec a}\vec b = -|\vec{SP}|$

$\vec{PS} = \text{proj}_{\vec a}\vec b$ is called the **vector projection of $\vec b$ onto $\vec a$**.

$|\vec{PS}| = \text{comp}_{\vec a}\vec b$ is called the **scalar projection of $\vec b$ onto $\vec a$** or the **component of $\vec b$ along $\vec a$**.

$$\boxed{\text{comp}_{\vec a}\vec b = \frac{\vec a\cdot\vec b}{|\vec a|}}$$

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vector projection of $\vec b$ onto $\vec a$

$\text{proj}_{\vec a}\vec b = |\vec{PS}|\cdot\dfrac{\vec a}{|\vec a|} = \dfrac{\vec a\cdot\vec b}{|\vec a|}\cdot\dfrac{\vec a}{|\vec a|} = \dfrac{\vec a\cdot\vec b}{|\vec a|^2}\vec a$

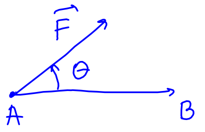
$$\boxed{\text{proj}_{\vec a}\vec b = \frac{\vec a\cdot\vec b}{|\vec a|^2}\vec a}$$

Example 5. Find the scalar and the vector projections of $\vec b = <4,2>$ onto $\vec a = \vec\imath + \vec\jmath$.

$= <1,1>$

$\text{comp}_{\vec a}\vec b = \dfrac{\vec a\cdot\vec b}{|\vec a|} = \dfrac{4+2}{\sqrt{1+1}} = \boxed{\dfrac{6}{\sqrt2}}$

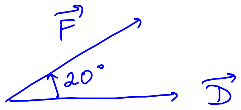
$\text{proj}_{\vec a}\vec b = \text{comp}_{\vec a}\vec b\cdot\dfrac{\vec a}{|\vec a|} = \dfrac{6}{\sqrt2}\cdot\dfrac{<1,1>}{\sqrt2} = \boxed{<3,3>}$



work done by the force

$W = \vec{F} \cdot \vec{AB}$

A woman exerts a horizontal force of 85 lb on a crate as she pushes it up a ramp that is 20 ft long and inclined at an angle of 20° above the horizontal. Find the work done on the box. (Give your answer correct to the nearest whole number.)



$|\vec{F}| = 85$ lb

$|\vec{D}| = 20$ ft

$W = \vec{F} \cdot \vec{D} = |\vec{F}| \cdot |\vec{D}| \cdot \cos\theta = (85)(20)\cos 20° \approx 1597$ (lb-ft)

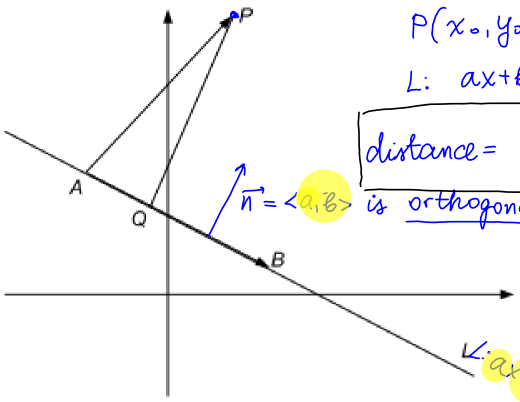
A force $\vec{F} = \langle 5, 8 \rangle$ moves an object from the point $(1,3)$ to the point $(11, 4)$. Find the work done by this force.

$\vec{D} = \langle 11-1, 4-3 \rangle = \langle 10, 1 \rangle$

$\vec{F} \cdot \vec{D} = \langle 5, 8 \rangle \cdot \langle 10, 1 \rangle = 50 + 8 = \boxed{58 \text{ lb-ft}}$

Definition. Given the nonzero vector $\vec{a} =< a_1, a_2 >$, the **orthogonal complement** of $\vec{a}$ is the vector $\vec{a}^{\perp} =< -a_2, a_1 >$.

Vectors $\vec{a}$ and $\vec{a}^{\perp}$ are orthogonal and $|\vec{a}| = |\vec{a}^{\perp}|$



$P(x_0, y_0)$

$L: \ ax + by + c = 0.$

$$\boxed{\text{distance} = \left| \frac{ax_0 + by_0 + c}{\sqrt{a^2+b^2}} \right|}$$

$\vec{n} = <a, b>$ is <u>orthogonal</u> to the line.

$L: ax + by + c = 0.$

The distance from the point P to the line L

$$\boxed{|PQ| = \text{comp}_{\overrightarrow{AB}^{\perp}} \overrightarrow{AP}}$$

Example 6. Find the distance from the point $(0, 4)$ to the line $2x + 5y = -3$.

$2x + 5y + 3 = 0$

$$D = \left| \frac{2(0) + 5(4) + 3}{\sqrt{2^2 + 5^2}} \right|$$