A cup of coffee has a temperature of 200°F and is in a room that has a temperature of 70°F. After 10 min the temperature of the coffee is 150°F.

(a) What is the temperature of the coffee after 15 min?

(b) When will the coffee have cooled to 100°F?

$$\frac{dT}{dt} = k(T-M), \quad \boxed{T(t) = M + (T(0)-M)e^{kt}}$$

$T(0) = 200$, $M = 70$

$T(10) = 150$, $\quad T(15) = ?$, $\quad$ if $T(t) = 100$, $t = ?$

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$$T(t) = 70 + 130e^{kt}$$

$$T(10) = 70 + 130e^{10k} = 150 \quad \text{solve for } k$$

$$130e^{10k} = 150 - 70$$

$$130e^{10k} = 80$$

$$e^{10k} = \frac{8}{13}$$

$$10k = \ln\frac{8}{13} \Rightarrow k = \frac{1}{10}\ln\frac{8}{13}$$

$$\boxed{T(t) = 70 + 130e^{\frac{t}{10}\ln\frac{8}{13}}}$$

$$T(15) = 70 + 130e^{\frac{15}{10}\ln\frac{8}{13}} = \boxed{\begin{array}{l} 70 + 130e^{\frac{3}{2}\ln\frac{8}{13}} \\ 70 + 130\left(\frac{8}{13}\right)^{3/2} \end{array}}$$

$$T(t) = 70 + 130e^{\frac{t}{10}\ln\frac{8}{13}} = 100$$

$$130e^{\frac{t}{10}\ln\frac{8}{13}} = 30$$

$$e^{\frac{t}{10}\ln\frac{8}{13}} = \frac{3}{13}$$

$$\frac{t}{10}\ln\frac{8}{13} = \ln\frac{3}{13}$$

$$\frac{t}{10} = \frac{\ln\frac{3}{13}}{\ln\frac{8}{13}}$$

$$\boxed{t = 10 \cdot \frac{\ln\frac{3}{13}}{\ln\frac{8}{13}}}$$

$= \langle 1, -2 \rangle$

1. Given vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = \langle -2, 3 \rangle$. Find

   (a) a unit vector $\mathbf{u}$ that has the same direction as $2\mathbf{b} + \mathbf{a}$.

   (b) angle between $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$

   (c) $\text{comp}_{\mathbf{b}}\mathbf{a}$, $\text{proj}_{\mathbf{b}}\mathbf{a}$.

(a) $2\vec{b}+\vec{a} = 2\langle -2, 3\rangle + \langle 1, -2\rangle$
$= \langle -4, 6\rangle + \langle 1, -2\rangle$
$= \langle -3, 4\rangle$

$|2\vec{b}+\vec{a}| = \sqrt{(-3)^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

$\vec{u} = \dfrac{2\vec{b}+\vec{a}}{|2\vec{b}+\vec{a}|} = \dfrac{\langle -3, 4\rangle}{5} = \boxed{\left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle}$

(b) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{\langle 1,-2\rangle \cdot \langle -2, 3\rangle}{\sqrt{1+4}\cdot\sqrt{4+9}} = \dfrac{-2-6}{\sqrt{5}\cdot\sqrt{13}} = \boxed{-\dfrac{8}{\sqrt{5}\cdot\sqrt{13}}}$

$\theta = \arccos\left(-\dfrac{8}{\sqrt{5}\cdot\sqrt{13}}\right)$

(c) $\text{comp}_{\vec{b}}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \boxed{\dfrac{-8}{\sqrt{13}}}$

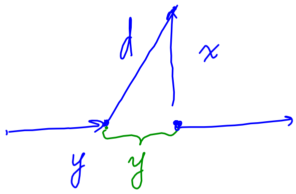
$\text{proj}_{\vec{b}}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\cdot\vec{b} = -\dfrac{8}{13}\langle -2, 3\rangle = \boxed{\left\langle \frac{16}{13}, -\frac{24}{13} \right\rangle}$

3. Find the distance from the point (-2,3) to the line $3x - 4y + 5 = 0$.

$P(x_0, y_0)$ $\quad ax+by+c=0.$ $\quad d = \left| \dfrac{ax_0 + by_0 + c}{\sqrt{a^2+b^2}} \right|$

$$d = \left| \frac{3(-2) - 4(3) + 5}{\sqrt{9+16}} \right| = \frac{13}{5}$$

14. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?




$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = +15$$

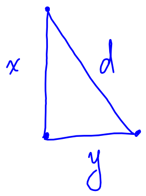
$$d^2 = x^2 + y^2$$

$d'(t) = ?$ when $t = 3$.

$$\frac{d}{dt}\left[d^2 = x^2 + y^2\right]$$

$$\cancel{2}d \cdot d'(t) = \cancel{2}x \cdot x'(t) + \cancel{2}y\, y'(t)$$

$$d'(t) = \frac{x \cdot x'(t) + y \cdot y'(t)}{d}$$



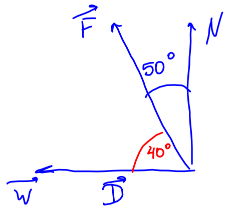

$$x = 45 + 3(5) = 60 = 4 \cdot 15$$

$$y = 3(15) = 45 = 3 \cdot 15$$

$$d = \sqrt{60^2 + 45^2} = 15\sqrt{4^2 + 3^2} = 15\sqrt{25} = 15(5) = 75$$

$$d'(t) = \boxed{\frac{60 \cdot 5 + 45(15)}{75}}$$

Find the work done by a force of 20 lb acting in the direction N50°W in moving an object 4 ft due west.



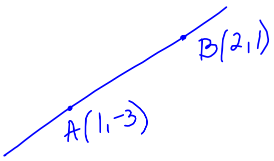

$|\vec{D}| = 4$

$|\vec{F}| = 20$

$\vec{W} = \vec{F} \cdot \vec{D} = |\vec{F}|\,|\vec{D}| \cos 40°$

$= 20(4) \cos 40°$

Find vector and parametric equations for the line passing through the points $A(1, -3)$ and $B(2, 1)$.




$\overrightarrow{AB} = \langle 2-1,\ 1-(-3) \rangle$

$= \langle 1, 4 \rangle$ parallel to the line.

vector equation $\vec{r}(t) = \langle 1, -3 \rangle + t \langle 1, 4 \rangle$

or $\vec{r}(t) = \langle 2, 1 \rangle + t \langle 1, 4 \rangle$

parametric equations: $\langle x, y \rangle = \langle 1, -3 \rangle + t \langle 1, 4 \rangle$

$\langle x, y \rangle = \langle 1+t, -3+4t \rangle$

$$\begin{cases} x = 1+t \\ y = -3+4t \end{cases}$$

7(b) $y = \dfrac{\sqrt[5]{2x-1}(x^2-4)^2}{\sqrt[3]{1+3x}}$

$$\ln y = \ln \frac{\sqrt[5]{2x-1}\,(x^2-4)^2}{\sqrt[3]{1+3x}} = \ln\left[\sqrt[5]{2x-1}\cdot(x^2-4)^2\right] - \ln\left[\sqrt[3]{1+3x}\right]$$

$$= \ln\left(\sqrt[5]{2x-1}\right) + \ln(x^2-4)^2 - \ln\left(\sqrt[3]{1+3x}\right)$$

$$= \ln(2x-1)^{1/5} + 2\ln(x^2-4) - \ln(1+3x)^{1/3}$$

$$= \frac{1}{5}\ln(2x-1) + 2\ln(x-2)(x+2) - \frac{1}{3}\ln(1+3x)$$

$$(\ln y)' = \left(\frac{1}{5}\ln(2x-1) + 2\ln(x-2) + 2\ln(x+2) - \frac{1}{3}\ln(1+3x)\right)'$$

$$\frac{1}{y}\cdot y' = \frac{1}{5}\,\frac{1}{2x-1}\,(2) + \frac{2}{x-2} + \frac{2}{x+2} - \frac{1}{\cancel{3}}\,\frac{1}{1+3x}\,(\cancel{3})$$

$$y' = y\left(\frac{2}{5}\,\frac{1}{2x-1} + \frac{2}{x-2} + \frac{2}{x+2} - \frac{1}{1+3x}\right)$$

$$\boxed{y' = \frac{\sqrt[5]{2x-1}\,(x^2-4)^2}{\sqrt[3]{1+3x}}\left(\frac{2}{5}\,\frac{1}{2x-1} + \frac{2}{x-2} + \frac{2}{x+2} - \frac{1}{1+3x}\right)}$$

13. Use differentials to estimate $(1.09)^{10}$.

$$\boxed{f(a+\Delta x) \approx f'(a)\,\Delta x + f(a)}$$

$a=1$, $\Delta x = 1.09-1 = 0.09$

$f(x)=x^{10}$, $f'(x) = 10x^9$

$f(1)=1$, $f'(1) = 10$

$$f(a+\Delta x) = (1.09)^{10} \approx f'(1)\,\Delta x + f(1)$$

$$= 10 \cdot 0.09 + 1$$

$$= \boxed{1.9}$$

15. Evaluate each limit:

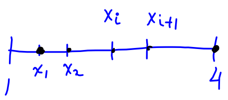
(a) $\lim_{x \to 0} \frac{\sin x + \sin 2x}{\sin 3x}$ $= \frac{0}{0} = \lim_{x \to 0} \frac{\cos x + 2\cos 2x}{3\cos 3x} = \frac{1+2}{3} = \boxed{1}$

7 (c) $y(t) = \sin^{-1} t$, $x(t) = \cos^{-1}(t^2)$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(\sin^{-1} t)'}{(\cos^{-1}(t^2))'} = \frac{\frac{1}{\sqrt{1-t^2}}}{-\frac{1}{\sqrt{1-(t^2)^2}}(2t)} = \boxed{-\frac{\sqrt{1-t^4}}{2t\sqrt{1-t^2}}}$$

smallest total area.

20. Find the area under the curve $y = x^2 + 3x - 2$ from 1 to 4. Use equal subintervals and take $x_i^*$ to be the right end-point of the $i$-th interval




$\Delta x = \frac{4-1}{n} = \frac{3}{n}$

Partition points: $x_0 = 1$

$x_1 = 1 + \Delta x = 1 + \frac{3}{n}$

$x_2 = 1 + 2\Delta x = 1 + 2 \cdot \frac{3}{n}$

$\cdots$

$x_{n-1} = 1 + (n-1) \cdot \frac{3}{n}$

$x_n = 4$

$x_i = 1 + i \cdot \frac{3}{n}$

$x_i^* = x_i, \quad i = 1, 2, 3, \ldots, n$

$$A \approx \sum_{i=1}^{n} f(x_i^*) \Delta x$$

$$\boxed{A \approx \sum_{i=1}^{n} \left[ \left(1 + \frac{3i}{n}\right)^2 + 3\left(1 + \frac{3i}{n}\right) - 2 \right] \frac{3}{n}}$$

$x_i^* = 1 + \frac{3i}{n}$

$$f(x_i^*) = \left(1 + \frac{3i}{n}\right)^2 + 3\left(1 + \frac{3i}{n}\right) - 2$$

21. Express the limit $\boxed{\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2}}$ as a definite integral. Do not evaluate it.

$$\int_a^b f(x)\,dx = \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x}$$

$\Delta x = \frac{1}{n}$ – integrating over an interval of length 1

$f(x_i^*) = \frac{1}{1+\left(\frac{i}{n}\right)^2}$

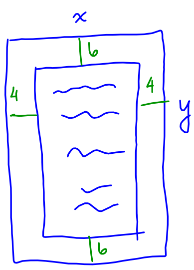
$x_i^* = \frac{i}{n}$ , $f(x) = \frac{1}{1+x^2}$ , $a=0$, $b=1$

$a = x_0 = \frac{0}{n} = 0$

$b = x_n = \frac{n}{n} = 1$

$$\boxed{\int_0^1 \frac{1}{1+x^2}\,dx}$$

19. The top and the bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of the printed material on the poster is fixed at 384 cm$^2$, find the dimensions of the poster with the smallest total area.




printed area: $(x-8)(y-12) = 384$ $\quad y = \frac{384}{x-8} + 12$

minimize $\quad A = xy$

$A = \frac{384x}{x-8} + 12x$ $\qquad 384 \lfloor 3 \quad 128$

$A' = 384 \cdot \frac{\cancel{x}-8-\cancel{x}}{(x-8)^2} + 12$

$= -8 \cdot 384 \cdot \frac{1}{(x-8)^2} + 12 = 0$

$12 = 8 \cdot 384 \cdot \frac{1}{(x-8)^2}$

$\frac{1}{(x-8)^2} = \frac{12}{8 \cdot 384}$

$(x-8)^2 = \frac{8 \cdot 384}{12} = \frac{2384}{3} = 256$

$(x-8)^2 = 256$

$x - 8 = 16, \quad \boxed{x = 24}$

$\boxed{y = \frac{384}{24-8} + 12 = 36}$

9. A particle moves on a vertical line so that its coordinate at time $t$ is $y = t^3 - 12t + 3$, $t \geq 0$.

(a) Find the velocity and acceleration functions.

(b) When is the particle moving upward? $(2, \infty)$

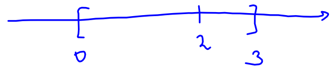
(c) Find the distance that particle travels in the time interval $0 \leq t \leq 3$

$v(t) = 3t^2 - 12$

$v(t) = 3t^2 - 12 > 0$

$t^2 > 4$, $\quad t > 2 \ (v > 0)$

$0 < t < 2 \ (v < 0)$




distance $= |y(2) - y(0)| + |y(3) - y(2)|$

$= |8 - 12(2) + 3 - 3| + |3^3 - 12(3) + 3 - 8 + 12(2) - 3|$

$= 16 + 7 = \boxed{25}$