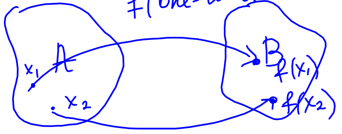
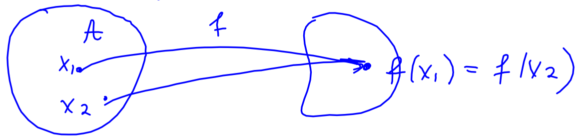
Section 1.5 **Inverse trigonometric functions.**

Definition. A function f with domain A is called **one-to-one function** if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

f (one-to-one)



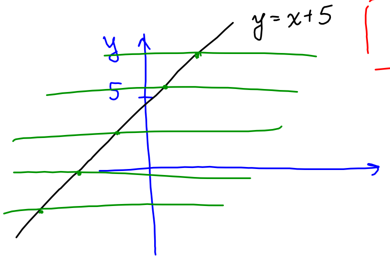
not one-to-one



Horizontal line test. A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 1. Determine which of the following functions is one-to-one:

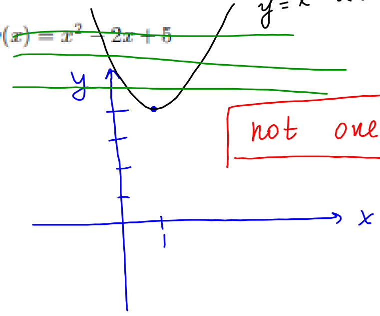
a) $f(x) = x + 5$



one-to-one

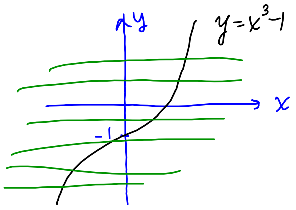
b) $g(x) = x^2 - 2x + 5$

$y = x^2 - 2x + 5$



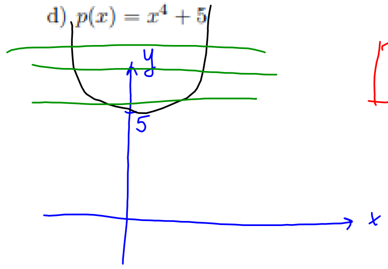
not one-to-one

c) $h(x) = x^3 - 1$



one-to-one

d) $p(x) = x^4 + 5$

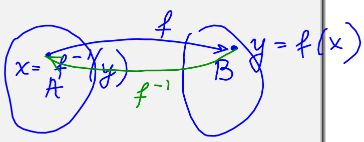


not one-to-one

Definition. Let f be one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

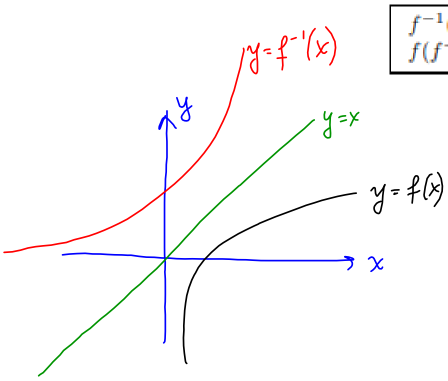


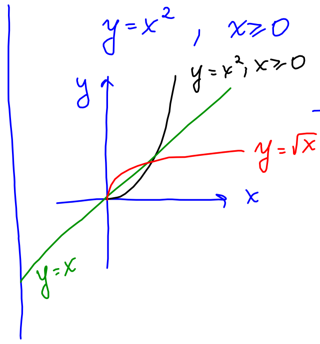
domain of f^{-1} = range of f
range of f^{-1} = domain of f

Let f be one-to-one function with domain A and range B. If $f(a) = b$, then $f^{-1}(b) = a$.

Cancellation equations

$$\boxed{\begin{array}{l} f^{-1}(f(x)) = x \text{ for every } x \in A \\ f(f^{-1}(x)) = x \text{ for every } x \in B \end{array}}$$





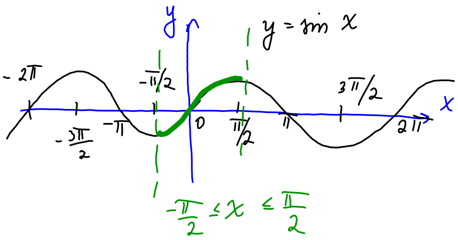
- not one-to-one on $(-\infty, \infty)$
- one-to-one on $[0, \infty)$

$y = \sqrt{x}$, $x \geq 0$

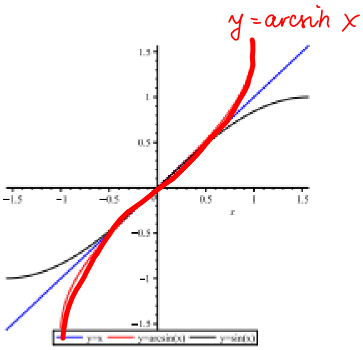
Inverse trigonometric functions

- **Inverse sine function**

$$\arcsin x = \sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$



DOMAIN	$-1 \le x \le 1$
RANGE	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



CANCELLATION EQUATIONS

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \quad \text{for } -1 \le x \le 1$$

Example 2. Find

1. $\sin^{-1}(0.5) = \frac{\pi}{6}$ because $\sin\frac{\pi}{6} = \frac{1}{2}$

2. $\sin^{-1}(\sin 1) = 1$

2

3. $\sin(2\overbrace{\sin^{-1}\tfrac{3}{5}}^{x}) = \sin 2x = 2\sin x\cos x = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \boxed{\frac{24}{25}}$

$x = \sin^{-1}\frac{3}{5} \Rightarrow \sin x = \frac{3}{5}$

$\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

4. $\arcsin(\sin\frac{5\pi}{4}) = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = \boxed{-\frac{\pi}{4}}$

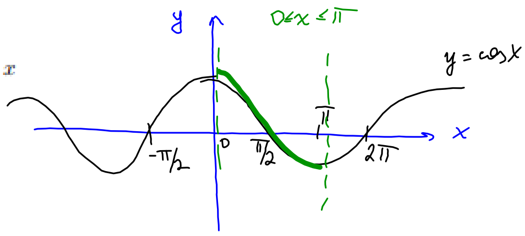
Example 3. Simplify $\tan(\underbrace{\sin^{-1}x}_{y}) = \tan y = \frac{\sin y}{\cos y} = \boxed{\frac{x}{\sqrt{1-x^2}}}$

$\sin^{-1}x = y \Rightarrow x = \sin y$

$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

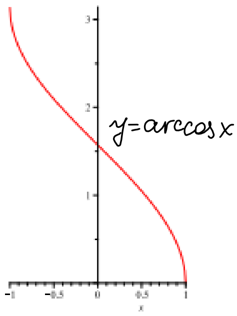
- **Inverse cosine function**

$$\arccos x = \cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$



DOMAIN $\quad -1 \le x \le 1$

RANGE $\quad 0 \le y \le \pi$



CANCELLATION EQUATIONS

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \le x \le \pi$$

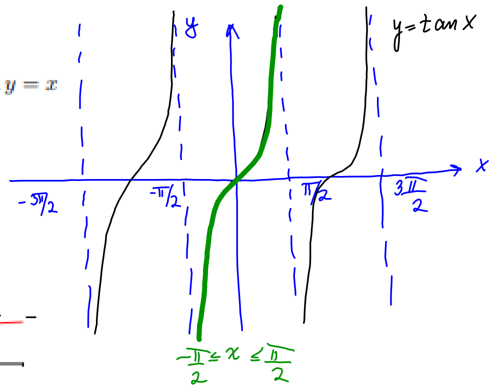
$$\cos(\cos^{-1} x) = x \quad \text{for } -1 \le x \le 1$$

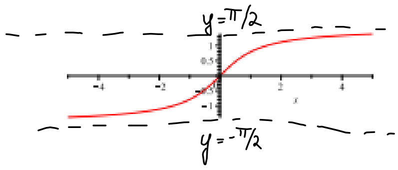
- **Inverse tangent function**

$$\arctan x = \tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

DOMAIN $\quad -\infty \leq x \leq \infty$

RANGE $\quad -\frac{\pi}{2} < y < \frac{\pi}{2}$





CANCELLATION EQUATIONS

$$\tan^{-1}(\tan x) = x \quad \text{for} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\tan(\tan^{-1} x) = x \quad \text{for} \quad -\infty \leq x \leq \infty$$

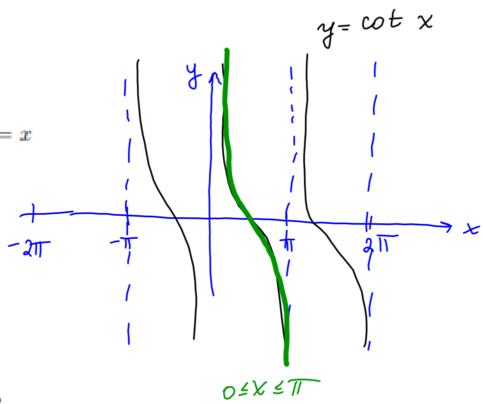
$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

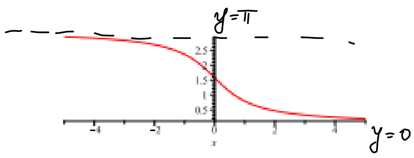
$$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

- **Inverse cotangent function**

$$\operatorname{arccot} x = \cot^{-1} x = y \quad \Leftrightarrow \quad \cot y = x$$

$$\boxed{\begin{array}{ll} \text{DOMAIN} & -\infty \le x \le \infty \\ \text{RANGE} & 0 < y < \pi \end{array}}$$





CANCELLATION EQUATIONS

$$\cot^{-1}(\cot x) = x \quad \text{for } 0 \le x \le \pi$$

$$\cot(\cot^{-1} x) = x \quad \text{for } -\infty \le x \le \infty$$

$$\lim_{x \to -\infty} \cot^{-1} x = 0$$

$$\lim_{x \to \infty} \cot^{-1} x = \pi$$

- **Other inverse trigonometric functions**

$$\csc^{-1} x = y \quad \Leftrightarrow \quad \csc y = x$$

DOMAIN $\quad |x| \geq 1$

RANGE $\quad y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$

$$\sec^{-1} x = y \quad \Leftrightarrow \quad \sec y = x$$

DOMAIN $\quad |x| \geq 1$

RANGE $\quad y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$