Chapter 2. **Limits and rates of change**
Section 2.2. **The limit of a function**

Definition. We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L" if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a but not equal to a.

Definition. We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-handed limit of** $f(x)$ **as** x **approaches** a **(or the limit of** $f(x)$ **as** x **approaches** a **from the left)**, equals L if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and $x < a$.

Definition. We write

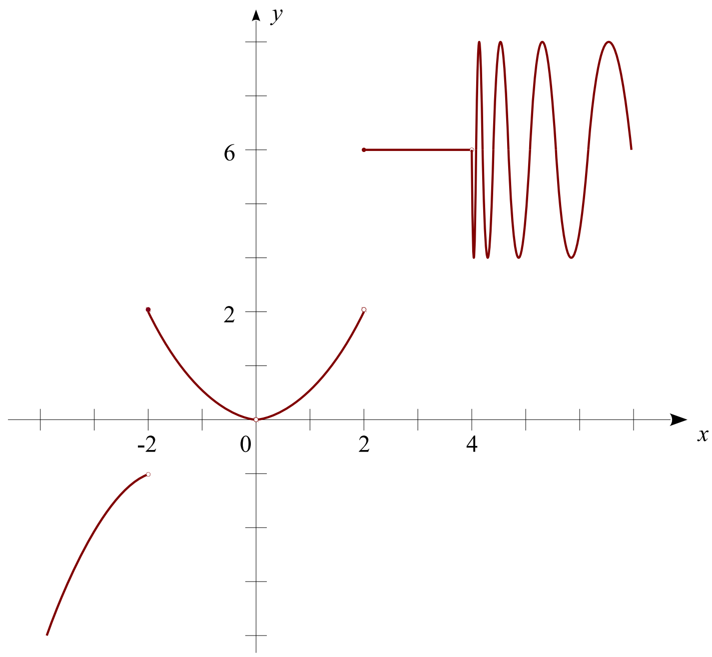
$$\lim_{x \to a^+} f(x) = L$$

and say the **right-handed limit of** $f(x)$ **as** x **approaches** a **(or the limit of** $f(x)$ **as** x **approaches** a **from the right)**, equals L if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and $x > a$.

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

Example 1. For the function g whose graph is given, state the value of the given quantity, if it exists.

1. $\lim_{x \to -2^-} g(x)$
2. $\lim_{x \to -2^+} g(x)$
3. $\lim_{x \to -2} g(x)$
4. $g(-2)$
5. $\lim_{x \to 0} g(x)$
6. $g(0)$
7. $\lim_{x \to 2^-} g(x)$
8. $\lim_{x \to 2^+} g(x)$
9. $g(2)$
10. $\lim_{x \to 4^-} g(x)$
11. $\lim_{x \to 4^+} g(x)$
12. $\lim_{x \to 4} g(x)$



Definition. Let f be a function defined on both sides of a, except, possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrary large by taking x to be sufficiently close to a but not equal to a.

Definition. Let f be a function defined on both sides of a, except, possibly at a itself. Then

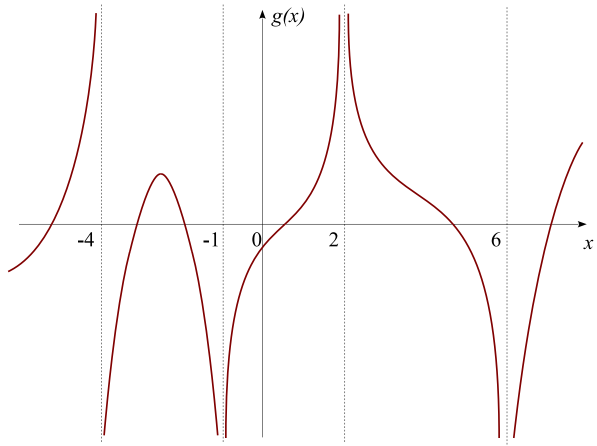
$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrary large negative by taking x to be sufficiently close to a but not equal to a.

Definition. The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$
$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

Example 2. For the function g whose graph is shown, state the following:



1. $\lim_{x\to -4} g(x)$
2. $\lim_{x\to -1} g(x)$
3. $\lim_{x\to 2} g(x)$
4. $\lim_{x\to 6} g(x)$

Example 3. Find

(a.) $\displaystyle\lim_{x\to 4^+} \frac{5}{x-4}$

(b.) $\displaystyle\lim_{x\to 4^-} \frac{5}{x-4}$

(c.) $\displaystyle\lim_{x\to 4} \frac{5}{x-4}$

Definition. We write

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{b}$$

and say "the limit of $\mathbf{r}(t)$, as t approaches a, equals $\mathbf{b}$" if we can make vector $\mathbf{r}(t)$ arbitrary close to $\mathbf{b}$ by taking t to be sufficiently close to a but not equal to a.

If $\mathbf{r}(t) =< f(t), g(t) >$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t) \right\rangle$$

provided the limits of the component functions exist.