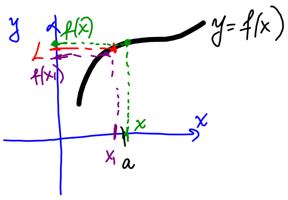
Chapter 2. **Limits and rates of change**
Section 2.2. **The limit of a function**

Definition. We write

$$\lim_{x \to a} f(x) = L$$

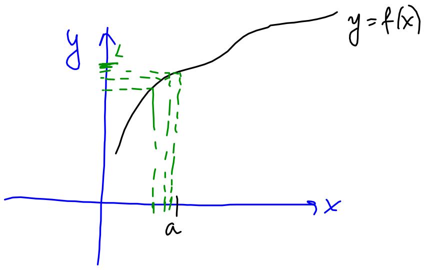
and say "the limit of $f(x)$, as x approaches a, equals L" if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a but not equal to a.



Definition. We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-handed limit of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the left)**, equals L if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and $x < a$.

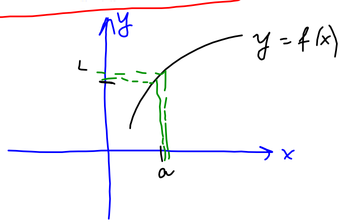


Definition. We write

$$\lim_{x \to a^+} f(x) = L$$

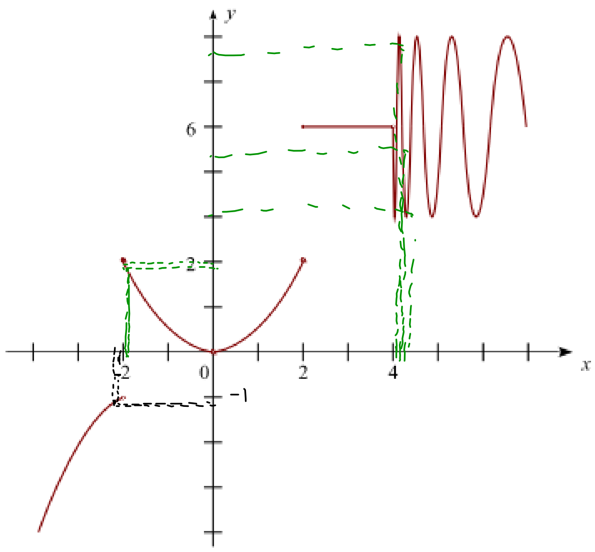
and say the **right-handed limit of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the right)**, equals L if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and $x > a$.

$$\boxed{\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L}$$

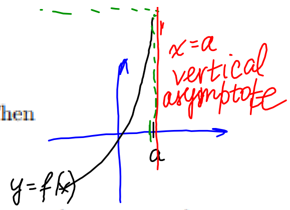


Example 1. For the function g whose graph is given, state the value of the given quantity, if it exists.

1. $\lim_{x \to -2^-} g(x)$ = -1
2. $\lim_{x \to -2^+} g(x)$ = 2
3. $\lim_{x \to -2} g(x)$ DNE
4. $g(-2)$ = 2
5. $\lim_{x \to 0} g(x)$ = 0
6. $g(0)$ DNE
7. $\lim_{x \to 2^-} g(x)$ = 2
8. $\lim_{x \to 2^+} g(x)$ = 6
9. $g(2)$ = 6
10. $\lim_{x \to 4^-} g(x)$ = 6
11. $\lim_{x \to 4^+} g(x)$ DNE
12. $\lim_{x \to 4} g(x)$ DNE



Definition. Let f be a function defined on both sides of a, except, possibly at a itself. Then



$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrary large by taking x to be sufficiently close to a but not equal to a.

Definition. Let f be a function defined on both sides of a, except, possibly at a itself. Then

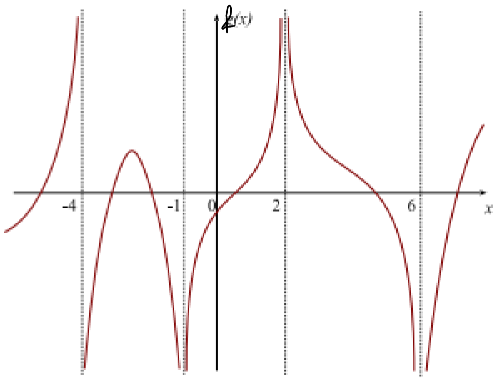
$$\lim_{x\to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrary large negative by taking x to be sufficiently close to a but not equal to a.

Definition. The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$\lim_{x\to a} f(x) = \infty$	$\lim_{x\to a+} f(x) = \infty$	$\lim_{x\to a^-} f(x) = \infty$
$\lim_{x\to a} f(x) = -\infty$	$\lim_{x\to a^+} f(x) = -\infty$	$\lim_{x\to a^-} f(x) = -\infty$

Example 2. For the function f whose graph is shown, state the following:



1. $\lim_{x \to -4} f(x)$ DNE $\quad \lim_{x \to -4^-} f(x) = \infty \quad \& \quad \lim_{x \to -4^+} f(x) = -\infty$
2. $\lim_{x \to -1} f(x) = -\infty$
3. $\lim_{x \to 2} f(x) = \infty$
4. $\lim_{x \to 6} f(x) = -\infty$

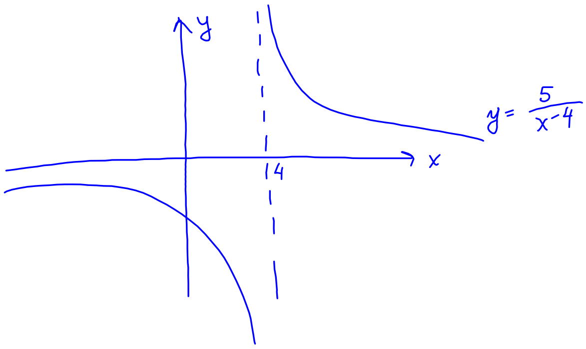
$x \to 6^-$

Example 3. Find

(a.) $\lim\limits_{x \to 4^+} \dfrac{5}{x-4} = \infty$

(b.) $\lim\limits_{x \to 4^-} \dfrac{5}{x-4} = -\infty$

(c.) $\lim\limits_{x \to 4} \dfrac{5}{x-4}$ DNE



Definition. We write

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{b}$$

and say "the limit of $\mathbf{r}(t)$, as t approaches a, equals $\mathbf{b}$" if we can make vector $\mathbf{r}(t)$ arbitrary close to $\mathbf{b}$ by taking t to be sufficiently close to a but not equal to a.

If $\mathbf{r}(t) =< f(t), g(t) >$, then

$$\boxed{\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t) \right\rangle}$$

provided the limits of the component functions exist.