Limit laws Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

Example 1. Given that $\lim_{x\to a} f(x) = 2$, $\lim_{x\to a} g(x) = -1$, and $\lim_{x\to a} h(x) = 10$. Find the limits that exist.

1. $\lim_{x\to a}[2f(x) - g(x) - h(x)]$

2. $\lim_{x\to a} \dfrac{g(x)}{h(x) - 2f(x)}$

Example 2. Evaluate the given limit and justify each step.

1. $\lim_{x\to 4}(2x^2 + 4x - 1)$

2. $\lim_{y \to 3} \frac{3(8y^2 - 1)}{2y^2(y-1)^4}$

3. $\lim_{x \to 3} \sqrt[4]{x^2 + 2x + 1}$

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$

Example 3. Evaluate each limit, if it exist.

1. $\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1}$

2. $\lim_{x \to -1} \frac{x^2 - x - 3}{x + 1}$

3. $\lim_{t \to 1} \frac{t^3 - t}{t^2 - 1}$

4. $\lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}}$

5. $\lim\limits_{x \to 0} \dfrac{x}{\sqrt{1+3x}-1}$

6. $\lim\limits_{t \to 2} \mathbf{r}(t)$, $\mathbf{r}(t) = \left\langle \dfrac{4-t}{2-\sqrt{t}}, \dfrac{t^2-4}{t-2} \right\rangle$

7. $\lim\limits_{x \to -3} |x+3|$

8. $\lim\limits_{x \to 2} \dfrac{|x-2|}{x-2}$

Example 4. Let

$$f(x) = \begin{cases} x^2 - 2x + 2, & \text{if } x < 1 \\ 3 - x, & \text{if } x \geq 1 \end{cases}$$

Find $\lim_{x \to 1} f(x)$.

Theorem If $f(x) \leq g(x)$ for all x in an open interval that contains a (except possibly at a) and the limits of f an g both exist as x approaches a, then

$$\boxed{\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)}$$

The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then

$$\boxed{\lim_{x \to a} g(x) = L}$$

Example 5. Use the Squeeze Theorem to show that $\lim_{x \to 0} x^2 \cos(20\pi x) = 0$.