Section 2.3 **Calculating limits using the limit laws**

Limit laws Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

Example 1. Given that $\lim_{x\to a} f(x) = 2$, $\lim_{x\to a} g(x) = -1$, and $\lim_{x\to a} h(x) = 10$. Find the limits that exist.

1. $\lim_{x\to a}[2f(x) - g(x) - h(x)] = \lim_{x\to a} 2f(x) - \lim_{x\to a} g(x) - \lim_{x\to a} h(x)$

   $= 2\lim_{x\to a} f(x) - \lim_{x\to a} g(x) - \lim_{x\to a} h(x)$

   $= 2(2) - (-1) - 10 = \boxed{-5}$

2. $\lim_{x\to a} \dfrac{g(x)}{h(x) - 2f(x)}$

   $= \dfrac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x) - 2\lim_{x\to a} f(x)} = \dfrac{-1}{10 - 2(2)} = \boxed{-\frac{1}{6}}$

Example 2. Evaluate the given limit and justify each step.

1. $\lim_{x\to 4}(2x^2 + 4x - 1) = 2(4^2) + 4(4) - 1 = 2(16) + 16 - 1 = \boxed{47}$

2. $\lim_{y \to 3} \frac{3(8y^2 - 1)}{2y^2(y-1)^4}$ $= \frac{3(8 \cdot (3^2) - 1)}{2(3^2)(3-1)^4}$ $= \frac{3(72-1)}{2(9)(16)}$ $= \boxed{\frac{71}{96}}$

3. $\lim_{x \to 3} \sqrt[4]{x^2 + 2x + 1}$ $= \sqrt[4]{3^2 + 2(3) + 1}$ $= \sqrt[4]{16}$ $= \boxed{2}$

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$

Example 3. Evaluate each limit, if it exist.

1. $\lim_{x\to -1} \frac{x^2-x-2}{x+1}$ $= \left|\frac{0}{0}\right| = \lim_{x\to -1} \frac{(x-2)\cancel{(x+1)}}{\cancel{x+1}} = \lim_{x\to -1} (x-2) = -1-2 = \boxed{-3}$

2. $\lim_{x\to -1} \frac{x^2-x-3}{x+1}$ $= \frac{1+1-3}{0} = \frac{-1}{0}$ $\boxed{DNE}$

3. $\lim_{t\to 1} \frac{t^3-t}{t^2-1}$ $= \frac{0}{0} = \lim_{t\to 1} \frac{t\cancel{(t^2-1)}}{\cancel{t^2-1}} = \lim_{t\to 1} t = \boxed{1}$

4. $\lim_{t\to 9} \frac{9-t}{3-\sqrt{t}}$ $= \frac{0}{0} = \lim_{t\to 9} \frac{(9-t)(3+\sqrt{t})}{(3-\sqrt{t})(3+\sqrt{t})} = \lim_{t\to 9} \frac{(9-t)(3+\sqrt{t})}{3^2-(\sqrt{t})^2} = \lim_{t\to 9} \frac{\cancel{(9-t)}(3+\sqrt{t})}{\cancel{9-t}}$

$= \lim_{t\to 9} (3+\sqrt{t}) = 3+\sqrt{9} = \boxed{6}$

5. $\lim_{x\to 0} \frac{x}{\sqrt{1+3x}-1} = \frac{0}{0} = \lim_{x\to 0} \frac{x(\sqrt{1+3x}+1)}{(\sqrt{1+3x}-1)(\sqrt{1+3x}+1)} = \lim_{x\to 0} \frac{x(\sqrt{1+3x}+1)}{(\sqrt{1+3x})^2-1^2}$

$$= \lim_{x\to 0} \frac{x(\sqrt{1+3x}+1)}{\cancel{1}+3x-\cancel{1}} = \lim_{x\to 0} \frac{\cancel{x}(\sqrt{1+3x}+1)}{3\cancel{x}} = \lim_{x\to 0} \frac{\sqrt{1+3x}+1}{3} = \boxed{\frac{2}{3}}$$

6. $\lim_{t\to 2} \mathbf{r}(t)$, $\mathbf{r}(t) = \left\langle \frac{4-t}{2-\sqrt{t}}, \frac{t^2-4}{t-2} \right\rangle$

$$= \left\langle \lim_{t\to 2} \frac{4-t}{2-\sqrt{t}}, \lim_{t\to 2} \frac{t^2-4}{t-2} \right\rangle = \left\langle \lim_{t\to 2} \frac{(4-t)(2+\sqrt{t})}{(2-\sqrt{t})(2+\sqrt{t})}, \lim_{t\to 2} \frac{\cancel{(t-2)}(t+2)}{\cancel{t-2}} \right\rangle$$

$$= \left\langle \lim_{t\to 2} \frac{\cancel{(4-t)}(2+\sqrt{t})}{\cancel{4-t}}, \lim_{t\to 2} (t+2) \right\rangle = \left\langle \lim_{t\to 2} (2+\sqrt{t}), 4 \right\rangle$$

$$= \boxed{\langle 2+\sqrt{2}, 4 \rangle}$$

7. $\lim_{x\to -3} |x+3| = \boxed{0}$

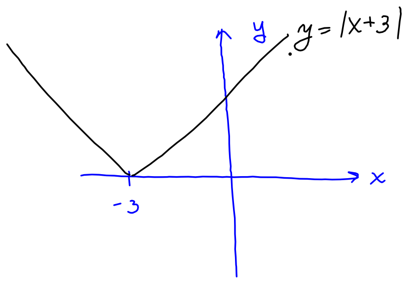
$$|x+3| = \begin{cases} x+3, & \text{if } x+3 \geq 0 \\ -(x+3), & \text{if } x+3 < 0 \end{cases}$$

$\lim_{x\to -3^+} |x+3| = \lim_{x\to -3} (x+3) = 0$

$x > -3$ or $x+3 > 0$

$\lim_{x\to -3^-} |x+3| = \lim_{x\to -3} (-(x+3)) = 0$

$x < -3$ or $x+3 < 0$



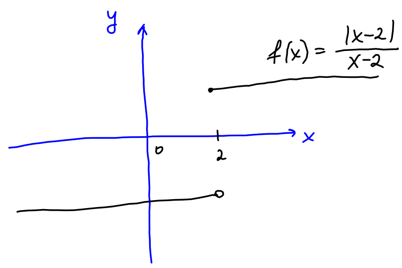
8. $\lim_{x\to 2} \frac{|x-2|}{x-2}$ $\boxed{\text{DNE}}$

$$|x-2| = \begin{cases} x-2, & \text{if } x-2 \geq 0 \\ -(x-2), & \text{if } x-2 < 0 \end{cases}$$

$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{\substack{x\to 2 \\ (x>2)}} \frac{x-2}{x-2} = \lim_{x\to 2} 1 = 1$

$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{\substack{x\to 2 \\ x<2}} \frac{-(x-2)}{x-2} = \lim_{x\to 2} (-1) = -1$

$\lim_{x\to 2^-} \neq \lim_{x\to 2^+}$



Example 4. Let

$$f(x) = \begin{cases} x^2 - 2x + 2, & \text{if } x < 1 \\ 3 - x, & \text{if } x \geq 1 \end{cases}$$

Find $\lim_{x\to 1} f(x)$. $\boxed{DNE}$

$$\lim_{x\to 1^-} f(x) = \lim_{\substack{x\to 1 \\ x<1}} f(x) = \lim_{x\to 1} (x^2-2x+2) = 1^2-2(1)+2 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{\substack{x\to 1 \\ x>1}} f(x) = \lim_{x\to 1} (3-x) = 3-1 = 2$$

$$2 \neq 1$$

Theorem If $f(x) \leq g(x)$ for all x in an open interval that contains a (except possibly at a) and the limits of f an g both exist as x approaches a, then

$$\boxed{\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)}$$

The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then

$$\boxed{\lim_{x\to a} g(x) = L}$$

Example 5. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^2 \cos(20\pi x) = 0$.

$$-1(x^2) \leq x^2 \cos 20\pi x \leq 1(x^2)$$

$$-x^2 \leq x^2 \cos 20\pi x \leq x^2$$

$$\lim_{x\to 0} x^2 = 0$$

$$\lim_{x\to 0} (-x^2) = 0$$

By the squeeze Theorem, $\lim_{x\to 0} x^2 \cos 20\pi x = 0$