**Definition.** A function $f$ is **continuous at a number** $a$ if

$$\boxed{\lim_{x\to a} f(x) = f(a)}$$

$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

If $f$ is not continuous at $a$, then $f$ has a **discontinuity** at $a$:

- if $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, then $f$ has a **jump discontinuity** at $a$,
- if either $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^-} f(x) = \infty$, then $f$ has an **infinity discontinuity** at $a$ and we say line $x = a$ is a **vertical asymptote** of the curve $y = f(x)$
- if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \neq f(a)$, then $f$ has a **removable discontinuity** at $a$

**Example 1.** Show that the function $f(x) = x^2 + 2x + 3$ is continuous at $a = 2$.

$\lim_{x\to 2} f(x) = \lim_{x\to 2} (x^2 + 2x + 3) = 2^2 + 2(2) + 3 = 11$

$f(2) = 2^2 + 2(2) + 3 = 11$

We've showed that

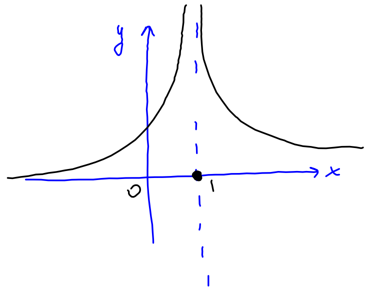
$\lim_{x\to 2} f(x) = f(2) = 11$

$f$ is continuous @ $a=2$.

**Example 2.** Explain why is the function

$$f(x) = \begin{cases} \dfrac{1}{(x-1)^2}, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$$

discontinuous at $a = 1$. Sketch the graph of the function.




$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$$

$$f(1) = 0$$

$\lim_{x \to 1} f(x) \neq f(1)$, so $f(x)$ is discontinuous @ $x=1$

infinite discontinuity

**Example 3.** Find the points at which $f$ is discontinuous.

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x < -1 \\ x, & \text{if } -1 \le x < 1 \\ \frac{1}{x^2}, & \text{if } x \ge 1 \end{cases}$$

check out the points $x = \pm 1$.

$\underline{x=-1.}$ $\lim\limits_{\substack{x\to -1^- \\ (x<-1)}} f(x) = \lim\limits_{x\to -1} \frac{1}{x} = -1$ | $f(-1) = -1$

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{\substack{x\to -1 \\ (x>-1)}} x = -1$

$f$ is continuous @ $x=-1$

$\underline{x=1}$ $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{\substack{x\to 1 \\ (x<1)}} x = 1$ | $f(1) = \frac{1}{1^2} = 1$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{\substack{x\to 1 \\ x>1}} \frac{1}{x^2} = 1$

$f$ is continuous @ $x=1$.

no points of discontinuity, $f$ is continuous on $(-\infty, \infty)$.

**Definition.** A function $f$ is **continuous from the right at a number** $a$ if

$$\lim_{x\to a^+} f(x) = f(a)$$

$f$ is **continuous from the left at a number** $a$ if

$$\lim_{x\to a^-} f(x) = f(a)$$

**Definition** A function $f$ is **continuous on an interval** if it is continuous at every number in the interval. (At an endpoint of the interval we understand **continuous** to mean **continuous from the right** or **continuous from the left**.)

**Example 4.** Show that the function $f(x) = x\sqrt{16 - x^2}$ is continuous on its domain. State the domain.

Domain : $16 - x^2 \geq 0$

$\underline{-4 \leq x \leq 4}$

Pick an arbitrary number $-4 \leq a \leq 4$

$$\lim_{x\to a} f(x) = \lim_{x\to a} x\sqrt{16-x^2} = a\sqrt{16-a^2}$$

$$f(a) = a\sqrt{16-a^2}$$

$$f(a) = \lim_{x\to a} f(x) \quad \text{for all } -4 \leq a \leq 4$$

This means, that $f(x)$ is continuous on $[-4, 4]$

**Example 5.** For what value of the constant $c$ is the function $f$ continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx + 1, & \text{if } x \le 3 \\ cx^2 - 1, & \text{if } x > 3 \end{cases}$$

$\underline{x=3}$

$$\lim_{x \to 3^-} f(x) = \lim_{\substack{x \to 3 \\ x < 3}} f(x) = \lim_{x \to 3} (cx+1) = 3c+1$$

$$\lim_{x \to 3^+} f(x) = \lim_{\substack{x \to 3 \\ x > 3}} f(x) = \lim_{x \to 3} (cx^2-1) = 9c-1$$

$$\underbrace{\lim_{x \to 3^-} f(x)}_{3c+1} = \underbrace{\lim_{x \to 3^+} f(x)}_{9c-1}$$

$$9c - 3c = 2 \quad \text{or} \quad 6c = 2 \quad \text{or} \quad \boxed{c = \tfrac{1}{3}}$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = 2$$

$$f(3) = c \cdot 3 + 1 = \tfrac{1}{3} \cdot 3 + 1 = 2$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

$c = \frac{1}{3}$ makes this function continuous everywhere.

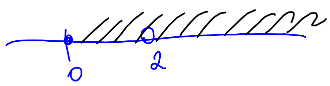
**Theorem.** If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following functions are also continuous at $a$:
1. $f + g$ 2. $f$ [redacted] $g$ 3. $cf$ 4. $fg$ 5. $\frac{f}{g}$ if $g(a) \neq 0$

**Theorem.**
(a.) Any polynomial is continuous on $(-\infty, \infty)$
(b.) Any rational function is continuous on its domain

**Theorem.** If $n$ is a positive even integer, then $f(x) = \sqrt[n]{x}$ is continuous on $[0, \infty)$. If $n$ is a positive odd integer, then $f$ is continuous on $(-\infty, \infty)$.

**Example 6.** On what interval is the function $h(x) = \sqrt{x} + \dfrac{1}{x-2} - \dfrac{1+2x}{x^2+4}$ continuous?

$\boxed{[0,2) \cup (2,\infty)}$




Domain for $\sqrt{x}$ is $[0, \infty)$

$\frac{1}{x-2}$ is discontinuous @ $x=2$

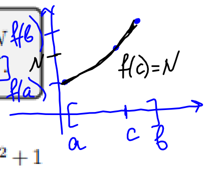
$\frac{1+2x}{x^2+4}$ is continuous for all $x$.

**Theorem.** If $f$ is continuous at $b$ and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b) = f\left(\lim_{x\to a} g(x)\right)$

$\lim_{x\to 4} \ln(x+4) = \ln\left[\lim_{x\to 4}(x+4)\right]$

**Theorem.** If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $(f\circ g)(x) = f(g(x))$ is continuous at $a$.

**The intermediate value theorem** Suppose that $f$ is continuous on the closed interval $[a, b]$ and let $N$ be any number strictly between $f(a)$ and $f(b)$. Then there exist a number $c$ in $(a, b)$ such that $f(c) = N$.




**Example 7.** Use the intermediate value theorem to show that there is a root of the equation $x^3 + 2x = x^2 + 1$ in the interval $(0,1)$.

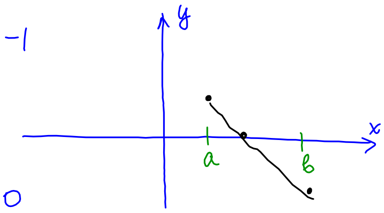
$x^3+2x = x^2+1$

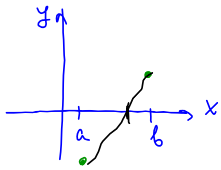
$f(x) = x^3+2x-x^2-1$

$f(0) = -1$

$f(1) = 1+2-1-1 = 1$

$f(0) < 0$ and $f(1) > 0$




By the Intemediate value theorem, there is a point $0<c<1$ such that $f(c)=0$

Then $c$ is a root of the given equation.