Definition Let f be a function defined on (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently large.

Definition Let f be a function defined on $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently large negative.

Definition The line $y = L$ is called a **horizontal asymptote of the curve** $y = f(x)$ if either

$$\boxed{\lim_{x \to \infty} f(x) = L}$$

or

$$\boxed{\lim_{x \to -\infty} f(x) = L}$$

Limit laws Suppose that c is a constant and the limits $\lim\limits_{x \to \pm\infty} f(x)$ and $\lim\limits_{x \to \pm\infty} g(x)$ exist. Then

1. $\lim\limits_{x \to \pm\infty} [f(x) + g(x)] = \lim\limits_{x \to \pm\infty} f(x) + \lim\limits_{x \to \pm\infty} g(x)$
2. $\lim\limits_{x \to \pm\infty} [f(x) - g(x)] = \lim\limits_{x \to \pm\infty} f(x) - \lim\limits_{x \to \pm\infty} g(x)$
3. $\lim\limits_{x \to \pm\infty} cf(x) = c \lim\limits_{x \to \pm\infty} f(x)$
4. $\lim\limits_{x \to \pm\infty} f(x)g(x) = \lim\limits_{x \to \pm\infty} f(x) \cdot \lim\limits_{x \to \pm\infty} g(x)$
5. $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to \pm\infty} f(x)}{\lim\limits_{x \to \pm\infty} g(x)}$ if $\lim\limits_{x \to \pm\infty} g(x) \neq 0$
6. $\lim\limits_{x \to \pm\infty} [f(x)]^n = \left[\lim\limits_{x \to \pm\infty} f(x)\right]^n$ where n is a positive integer
7. $\lim\limits_{x \to \pm\infty} c = c$
8. $\lim\limits_{x \to \pm\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to \pm\infty} f(x)}$ where n is a positive integer

Theorem If $r > 0$ is a rational number, then

$$\boxed{\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \frac{1}{x^r} = 0}$$

Example 1. Find each of the following limits:

1. $\displaystyle\lim_{y\to\infty} \frac{7y^3 + 4y}{2y^3 - y^2 + 3}$

2. $\displaystyle\lim_{x\to\infty} \frac{x + 4}{x^3 - 3}$

3. $\displaystyle\lim_{t\to-\infty} \frac{t^2 - 3t + 1}{2t + 3}$

4. $\displaystyle\lim_{t\to\infty} \frac{t^5 - 5t + 151}{2018 - t^3}$

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \begin{cases} \dfrac{a_n}{b_m}, & \text{if } n = m \\ 0, & \text{if } n < m \\ \infty, & \text{if } n > m \text{ and } \dfrac{a_n}{b_m} > 0 \\ -\infty, & \text{if } n > m \text{ and } \dfrac{a_n}{b_m} < 0 \end{cases}$$

Example 2. Evaluate the following limits:

1. $\lim_{x\to\infty} \frac{\sqrt{x^2+4x}}{4x+1}$

2. $\lim_{x\to-\infty} \frac{\sqrt{4x^6-x}}{x^3+1}$

3. $\lim_{x\to\infty} \sin x$

4. $\lim_{x\to\infty} \sin\frac{1}{x}$

5. $\lim_{x\to\infty} \frac{\sin^2 x}{x^2}$

6. $\lim_{x\to\infty} (\sqrt{x^2+3x+1}-x)$

7. $\lim_{x \to -\infty} (x + \sqrt{x^2 + 6x})$

8. $\lim_{x \to \infty} \frac{3}{15 + e^{2x}}$

9. $\lim_{x \to -\infty} \frac{3}{15 + e^{2x}}$

10. $\lim_{x \to -\infty} \frac{2^{3x} - 2^{-3x}}{2^{3x} + 2^{-3x}}$

11. $\lim_{x \to \infty} \frac{2^{3x} - 2^{-3x}}{2^{3x} + 2^{-3x}}$

12. $\lim_{x\to\infty} [\ln(2+x) - \ln(3+x)]$

Example 3. Find the horizontal and vertical asymptotes of each curve

1. $y = \dfrac{x^2+4}{x^2-1}$

2. $y = \dfrac{x^3+1}{x^2+x}$