Definition Let f be a function defined on (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently large.

Definition Let f be a function defined on $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently large negative.

Definition The line $y = L$ is called a **horizontal asymptote of the curve** $y = f(x)$ if either

$$\boxed{\lim_{x\to\infty} f(x) = L}$$

or

$$\boxed{\lim_{x\to-\infty} f(x) = L}$$

Limit laws Suppose that c is a constant and the limits $\lim\limits_{x\to\pm\infty} f(x)$ and $\lim\limits_{x\to\pm\infty} g(x)$ exist. Then

1. $\lim\limits_{x\to\pm\infty}[f(x)+g(x)] = \lim\limits_{x\to\pm\infty} f(x) + \lim\limits_{x\to\pm\infty} g(x)$
2. $\lim\limits_{x\to\pm\infty}[f(x)-g(x)] = \lim\limits_{x\to\pm\infty} f(x) - \lim\limits_{x\to\pm\infty} g(x)$
3. $\lim\limits_{x\to\pm\infty} cf(x) = c\lim\limits_{x\to\pm\infty} f(x)$
4. $\lim\limits_{x\to\pm\infty} f(x)g(x) = \lim\limits_{x\to\pm\infty} f(x)\cdot\lim\limits_{x\to\pm\infty} g(x)$
5. $\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to\pm\infty} f(x)}{\lim\limits_{x\to\pm\infty} g(x)}$ if $\lim\limits_{x\to\pm\infty} g(x) \neq 0$
6. $\lim\limits_{x\to\pm\infty}[f(x)]^n = \left[\lim\limits_{x\to\pm\infty} f(x)\right]^n$ where n is a positive integer
7. $\lim\limits_{x\to\pm\infty} c = c$
8. $\lim\limits_{x\to\pm\infty}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to\pm\infty} f(x)}$ where n is a positive integer

Theorem If $r > 0$ is a rational number, then

$$\boxed{\lim_{x\to\infty}\frac{1}{x^r} = \lim_{x\to-\infty}\frac{1}{x^r} = 0}$$

Example 1. Find each of the following limits:

1. $\lim\limits_{y\to\infty}\dfrac{7y^3+4y}{2y^3-y^2+3} = \lim\limits_{y\to\infty}\dfrac{\cancel{y^3}\left[7+\frac{4y}{y^3}\right]}{\cancel{y^3}\left[2-\frac{y^2}{y^3}+\frac{3}{y^3}\right]} = \lim\limits_{y\to\infty}\dfrac{7+\frac{4}{y^2}}{2-\frac{1}{y}+\frac{3}{y^3}} = \boxed{\dfrac{7}{2}}$

2. $\lim\limits_{x\to\infty}\dfrac{x+4}{x^3-3} = \lim\limits_{x\to\infty}\dfrac{x\left(1+\frac{4}{x}\right)}{x^3\left(1-\frac{3}{x^3}\right)} = \lim\limits_{x\to\infty}\dfrac{1}{x^2} = \boxed{0}$

3. $\lim\limits_{t\to-\infty}\dfrac{t^2-3t+1}{2t+3} = \lim\limits_{t\to-\infty}\dfrac{t^{\cancel{2}}\left(1-\frac{3t}{t^2}+\frac{1}{t^2}\right)}{\cancel{t}\left(2+\frac{3}{t}\right)} = \lim\limits_{t\to-\infty}\dfrac{t}{2} = \boxed{-\infty}$

4. $\lim\limits_{t\to\infty}\dfrac{t^5-5t+151}{2018-t^3} = \lim\limits_{t\to\infty}\dfrac{t^5\left(1+\frac{5t}{t^4}+\frac{151}{t^5}\right)}{\left(\frac{2018}{t^3}-1\right)t^3} = \lim\limits_{t\to\infty}\dfrac{t^5}{-t^3} = \lim\limits_{t\to\infty}(-t^2) = \boxed{-\infty}$

$$\lim_{x\to\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\ldots+b_1x+b_0} = \begin{cases}\frac{a_n}{b_m}, & \text{if } n=m\\ 0, & \text{if } n<m\\ \infty, & \text{if } n>m \text{ and } \frac{a_n}{b_m}>0\\ -\infty, & \text{if } n>m \text{ and } \frac{a_n}{b_m}<0\end{cases}$$

$\sqrt{x^2} = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

Example 2. Evaluate the following limits:

1. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+4x}}{4x+1} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^2(1+\cancel{\frac{4}{x}}^{0})}}{x(4+\cancel{\frac{1}{x}}^{0})} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^2}}{4x} = \lim\limits_{x\to\infty} \dfrac{x}{4x} = \boxed{\dfrac{1}{4}}$

$\sqrt{x^6} = -x^3$ if $x < 0$

2. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{4x^6-x}}{x^3+1} = \lim\limits_{x\to-\infty} \dfrac{\sqrt{x^6(4-\cancel{\frac{x}{x^6}}^{0})}}{x^3(1+\cancel{\frac{1}{x^3}}^{0})} = \lim\limits_{x\to-\infty} \dfrac{\sqrt{4x^6}}{x^3} = \lim\limits_{x\to-\infty} \dfrac{-2\cancel{x^3}}{\cancel{x^3}} = \boxed{-2}$

3. $\lim\limits_{x\to\infty} \sin x$ DNE

4. $\lim\limits_{x\to\infty} \sin\dfrac{1}{x} = \sin\left(\lim\limits_{x\to\infty} \dfrac{1}{x}\right) = \sin 0 = \boxed{0}$

$\sin x$ is continuous

5. $\lim\limits_{x\to\infty} \dfrac{\sin^2 x}{x^2}$

Do the Squeeze Theorem.

$0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2} \Rightarrow \underbrace{0}_{} \le \dfrac{\sin^2 x}{x^2} \le \underbrace{\dfrac{1}{x^2}}_{}$

$\lim\limits_{x\to\infty} 0 = 0$, $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$. By the Squeeze Thm, $\lim\limits_{x\to\infty} \dfrac{\sin^2 x}{x^2} = \boxed{0}$

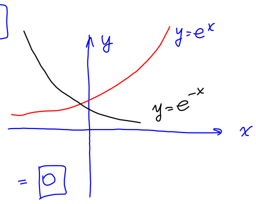
6. $\lim\limits_{x\to\infty} (\sqrt{x^2+3x+1} - x)\dfrac{(\sqrt{x^2+3x+1} + x)}{\sqrt{x^2+3x+1} + x} = \lim\limits_{x\to\infty} \dfrac{(\sqrt{x^2+3x+1})^2 - x^2}{\sqrt{x^2+3x+1} + x}$

$= \lim\limits_{x\to\infty} \dfrac{\cancel{x^2}+3x+1-\cancel{x^2}}{\sqrt{x^2(1+\cancel{\frac{3}{x}}^{0}+\cancel{\frac{1}{x^2}}^{0})}+x} = \lim\limits_{x\to\infty} \dfrac{3x+1}{\sqrt{x^2}+x} = \lim\limits_{x\to\infty} \dfrac{3x+1}{2x} = \boxed{\dfrac{3}{2}}$

$\sqrt{x^2} = |x|$, $\sqrt{x^2} = -x$, if $x < 0$

7. $\lim\limits_{x\to-\infty} (x+\sqrt{x^2+6x})\dfrac{(x-\sqrt{x^2+6x})}{x-\sqrt{x^2+6x}} = \lim\limits_{x\to-\infty} \dfrac{x^2-(\sqrt{x^2+6x})^2}{x-\sqrt{x^2(1+\cancel{\frac{6}{x}}^{0})}} = \lim\limits_{x\to-\infty} \dfrac{x^2-x^2-6x}{x-\cancel{\sqrt{x^2}}\,-x}$

$= \lim\limits_{x\to-\infty} \dfrac{-6x}{2x} = \boxed{-3}$



$\lim\limits_{x\to\infty} e^x = \infty$
$\lim\limits_{x\to-\infty} e^x = 0$
$\lim\limits_{x\to\infty} e^{-x} = 0$
$\lim\limits_{x\to-\infty} e^{-x} = \infty$

8. $\lim\limits_{x\to\infty} \dfrac{3}{15+e^{2x}} = \dfrac{3}{15+\cancel{\lim\limits_{x\to\infty} e^{2x}}\,\infty} = \boxed{0}$

9. $\lim\limits_{x\to-\infty} \dfrac{3}{15+\cancel{e^{2x}}^{0}} = \dfrac{3}{15} = \boxed{\dfrac{1}{5}}$

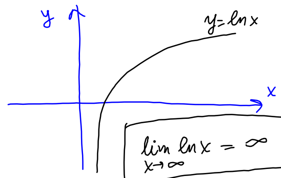
10. $\lim\limits_{x\to-\infty} \dfrac{2^{3x}-2^{-3x}}{2^{3x}+2^{-3x}} = \lim\limits_{x\to-\infty} \dfrac{\cancel{2^{-3x}}\left(\frac{2^{3x}}{2^{-3x}}-1\right)}{\cancel{2^{-3x}}\left(\frac{2^{3x}}{2^{-3x}}+1\right)} = \lim\limits_{x\to-\infty} \dfrac{2^{3x-(-3x)}-1}{2^{3x-(-3x)}+1}$

$= \lim\limits_{x\to-\infty} \dfrac{\cancel{2^{6x}}^{0}-1}{\cancel{2^{6x}}^{0}+1} = \boxed{-1}$

11. $\lim\limits_{x\to\infty} \dfrac{2^{3x}-2^{-3x}}{2^{3x}+2^{-3x}} = \lim\limits_{x\to\infty} \dfrac{\cancel{2^{3x}}\left(1-\frac{2^{-3x}}{2^{3x}}\right)}{\cancel{2^{3x}}\left(1+\frac{2^{-3x}}{2^{3x}}\right)} = \lim\limits_{x\to\infty} \dfrac{1-\cancel{2^{-6x}}^{0}}{1+\cancel{2^{-6x}}^{0}} = \boxed{1}$

12. $\lim_{x\to\infty}[\ln(2+x)-\ln(3+x)] = \lim_{x\to\infty} \ln\frac{2+x}{3+x}$

$= \ln\left(\lim_{x\to\infty}\frac{2+x}{3+x}\right) = \ln\frac{2}{3}$



$\boxed{\lim_{x\to\infty}\ln x = \infty}$

$\boxed{\lim_{x\to 0^+}\ln x = -\infty}$

Example 3. Find the horizontal and vertical asymptotes of each curve

1. $y=\frac{x^2+4}{x^2-1}$

$x^2-1=0 \Rightarrow \boxed{x=1,\ x=-1 \text{ - vertical asymptotes}}$

horizonal asymptotes:

$\lim_{x\to\infty} y = \lim_{x\to\infty}\frac{x^2+4}{x^2-1} = 1$

$\boxed{\text{horizontal asymptote: } y=1}$

$\boxed{a^3+b^3=(a+b)(a^2+ab+b^2)}$

2. $y=\frac{x^3+1}{x^2+x} = \frac{\cancel{(x+1)}(x^2+x+1)}{x\cancel{(x+1)}} = \frac{x^2+x+1}{x}$

$\boxed{\text{Vertical asymptote } x=0}$

$\lim_{x\to\infty}\frac{x^2+x+1}{x} = \infty$

$\boxed{\text{no horizontal asymptotes}}$

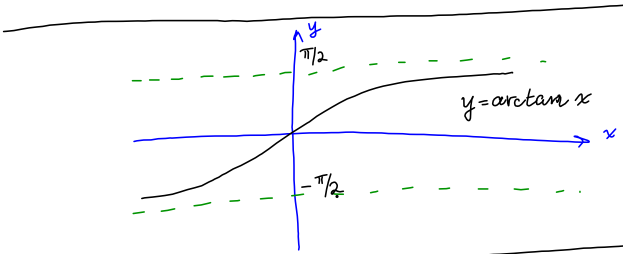
E.2 continued

$\lim_{t\to0^+}\ln t = -\infty$

13. $\lim_{x\to\infty}\left[\ln(x-4)-\ln(x^3-3)\right] = \lim_{x\to\infty}\ln\frac{x-4}{x^3-3} = \ln\left(\overset{0}{\lim_{x\to\infty}\frac{x-4}{x^3-3}}\right) = \boxed{-\infty}$

note, that $\frac{x-4}{x^3-3}\to 0$.

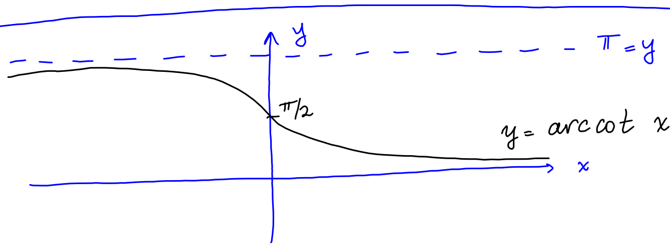
14. $\lim_{x\to\infty}\left[\ln(x^3-4)-\ln(x^2+1)\right] = \lim_{x\to\infty}\ln\frac{x^3-4}{x^2+1} = \ln\left[\lim_{x\to\infty}\frac{x^3-4}{x^2+1}\right] = \lim_{t\to\infty}\ln t = \boxed{\infty}$



$\boxed{\lim_{x\to\infty}\arctan x = \frac{\pi}{2}}$

$\boxed{\lim_{x\to-\infty}\arctan x = -\frac{\pi}{2}}$

15. $\lim_{x\to\infty}\arctan\frac{x^5-4}{4-x^3} = \arctan\left(\overset{-\infty}{\lim_{x\to\infty}\frac{x^5-4}{4-x^3}}\right) = \lim_{t\to-\infty}\arctan t = \boxed{-\frac{\pi}{2}}$



$\boxed{\lim_{x\to\infty}\operatorname{arccot} x = 0}$

$\boxed{\lim_{x\to-\infty}\operatorname{arccot} x = \pi}$