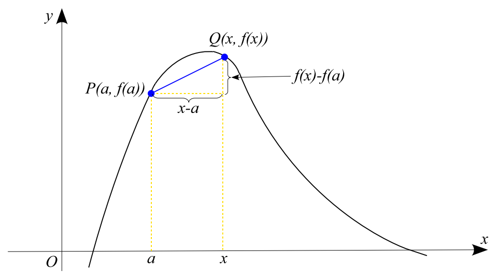
Section 2.7 **Derivatives and rates of change**

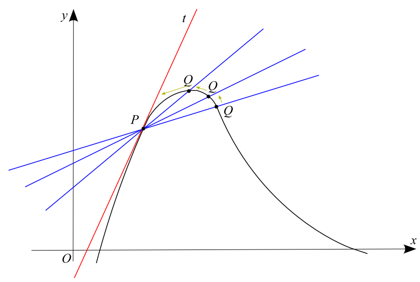
The tangent line.

If a curve C has equation $y = f(x)$ and we want to find the tangent to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$



Then we let Q approach P along the curve C by letting x approach a.



If m_{PQ} approaches a number m, then we define the **tangent** t to be the line through P with slope m.

Definition. The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

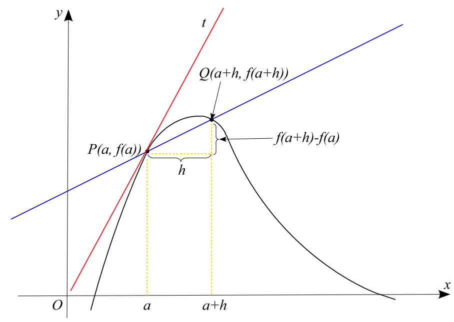
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists. Then the equation of the tangent line is

$$y = m(x - a) + f(a)$$

Let $h = x - a$, then $x = a + h$, so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$



Then the slope of the tangent line becomes

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 1. Find the equation of the tangent line to the curve $y = \sqrt{2x-3}$ at the point (2,1).

Velocity.

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement of the object from the origin at time t. Function f is called the **position function** of the object.

$$\text{average velocity } = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}$$

Then the **velocity** or **instantaneous velocity** at time $t = a$ is

$$v(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Example 2. The displacement of an object moving in a straight line is given by $s(t) = 1 + 2t + t^2/4$ (t is in seconds).

(a) Find the average velocity over the time period [1,3]

(b) Find the instantaneous velocity when $t = 1$

Example 3. The object is moving upward. Its height after t sec is given by $h(t) = 58t - 0.83t^2$

(a) What is the maximum height reached by the object?

(b) Find the instantaneous velocity at $t = 1$

Definition. The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if the limit exist.

Example 1. Find $f'(a)$ if $f(x) = x^2 + 3x - 1$.

Geometric interpretation of the derivative. $f'(a)$ is the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$.

Example 2. Find an equation of the tangent line to $f(x) = x^2 + 3x - 1$ at the point (-1,-3).

Other interpretations of the derivative.

- $f'(a)$ is the instanteneous rate of change of $y = f(x)$ with respect to x when $x = a$.
- if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the velocity of the particle at time $t = a$

Other rates of change.

Suppose y is a quantity that depends on another quantity x or $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta x = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of** y **with respect to** x over the interval $[x_1, x_2]$.

The **instantaneous rate of change of** y **with respect to** x at $x = x_1$ is equal to

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1)$$