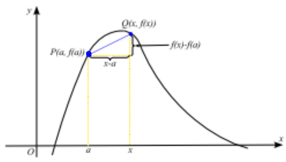
Section 2.7 **Derivatives and rates of change**

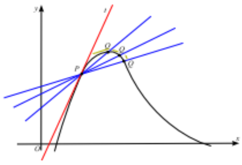
The tangent line.

If a curve C has equation $y = f(x)$ and we want to find the tangent to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$ slope of the secant line



Then we let Q approach P along the curve C by letting x approach a.



If m_{PQ} approaches a number m, then we define the **tangent** t to be the line through P with slope m.

Definition. The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$\boxed{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

provided that this limit exists. Then the equation of the tangent line is

$$\boxed{y = m(x - a) + f(a)}$$

Let $h = x - a$, then $x = a + h$, so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

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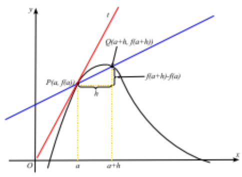

Then the slope of the tangent line becomes

$$\boxed{m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}}$$ slope of the tangent line.

Example 1. Find the equation of the tangent line to the curve $y = \sqrt{2x-3}$ at the point $(2, 1)$.

$$y = m(x-2)+1$$

$$m = \lim_{x\to 2} \frac{y(x)-y(2)}{x-2} = \lim_{x\to 2} \frac{(\sqrt{2x-3}-1)(\sqrt{2x-3}+1)}{(x-2)(\sqrt{2x-3}+1)}$$

$$= \lim_{x\to 2} \frac{2x-3-1}{(x-2)(\sqrt{2x-3}+1)} = \lim_{x\to 2} \frac{2x-4}{(x-2)(\sqrt{2x-3}+1)}$$

$$= \lim_{x\to 2} \frac{2\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{2x-3}+1)} = \frac{2}{\sqrt{4-3}+1} = 1$$

$\boxed{y = 1(x-2)+1}$ tangent line.

Velocity.

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement of the object from the origin at time t. Function f is called the **position function** of the object.

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}$$

Then the **velocity** or **instantaneous velocity** at time $t = a$ is

$$v(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Example 2. The displacement of an object moving in a straight line is given by $s(t) = 1 + 2t + t^2/4$ (t is in seconds).

(a) Find the average velocity over the time period [1,3]

$$\text{average velocity} = \frac{s(3)-s(1)}{3-1} = \frac{\overbrace{\cancel{1}+2(3)+\frac{3^2}{4}}^{s(3)} - (\overbrace{\cancel{1}+2(1)+\frac{1}{4}}^{s(1)})}{2}$$

$$= \frac{6-2+\frac{9}{4}-\frac{1}{4}}{2} = 3$$

(b) Find the instantaneous velocity when $t = 1$

$$v = \lim_{t\to 1} \frac{s(t)-s(1)}{t-1} = \lim_{t\to 1} \frac{\cancel{1}+2t+\frac{t^2}{4} - (\overbrace{\cancel{1}+2+\frac{1}{4}}^{s(1)})}{t-1}$$

$$= \lim_{t\to 1} \frac{2t-2+\frac{t^2}{4}-\frac{1}{4}}{t-1} = \lim_{t\to 1} \frac{2(t-1)+\frac{1}{4}(t^2-1)}{t-1}$$

$$= \lim_{t\to 1} \frac{2(t-1)+\frac{1}{4}(t-1)(t+1)}{t-1} = \lim_{t\to 1} \frac{\cancel{(t-1)}[2+\frac{1}{4}(t+1)]}{\cancel{t-1}} = 2+\frac{1}{4}(1+1) = \boxed{\frac{5}{2}}$$

Example 3. The object is moving upward. Its height after t sec is given by $h(t) = 58t - 0.83t^2$

(a) What is the maximum height reached by the object?

$$v = \lim_{a\to 0} \frac{h(t+a)-h(t)}{a} = \lim_{a\to 0} \frac{58(t+a) - 0.83(t+a)^2 - (58t - 0.83t^2)}{a}$$

$$= \lim_{a\to 0} \frac{58a - 0.83[\cancel{t^2}+2at+a^2-\cancel{t^2}]}{a} = \lim_{a\to 0} \frac{58a - 0.83(2at+a^2)}{a}$$

$$= \lim_{a\to 0} \frac{\cancel{a}[58-0.83(2t+a)]}{\cancel{a}} = 58-0.83(2t) = 0 \Rightarrow t = \frac{58}{1.66}$$

$$\boxed{\text{max height} = h\left(\frac{58}{1.66}\right) = 58 \cdot \frac{58}{1.66} - 0.83\left(\frac{58}{1.66}\right)^2}$$

(b) Find the instantaneous velocity at $t = 1$

$v(t) = 58 - 1.66t$

$\boxed{v(1) = 58 - 1.66}$

Definition. The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

if the limit exist.

Example 1. Find $f'(a)$ if $f(x) = x^2 + 3x - 1$.

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\overbrace{x^2+3x-1}^{f(x)} - \overbrace{(a^2+3a-1)}^{f(a)}}{x-a}$$

$$= \lim_{x\to a} \frac{x^2+3x-a^2-3a}{x-a} = \lim_{x\to a} \frac{(x^2-a^2)+(3x-3a)}{x-a}$$

$$= \lim_{x\to a} \frac{(x-a)(x+a)+3(x-a)}{x-a} = \lim_{x\to a} \frac{\cancel{(x-a)}[x+a+3]}{\cancel{x-a}}$$

$$= \lim_{x\to a} (x+a+3) = 2a+3$$

$$\boxed{f'(a) = 2a+3}$$

an equation of the tangent line to $y=f(x)$ @ $x=a$ is

$$y - f(a) = f'(a)(x-a)$$

Geometric interpretation of the derivative. $f'(a)$ is the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$.

Example 2. Find an equation of the tangent line to $f(x) = x^2 + 3x - 1$ at the point (-1,-3).

slope $= f'(-1) = 2(-1)+3 = 1$

tangent line

$$y-(-3) = 1(x-(-1))$$

$$\boxed{y+3 = x+1}$$

Other interpretations of the derivative.

- $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.
- if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the velocity of the particle at time $t = a$

Other rates of change.

Suppose y is a quantity that depends on another quantity x or $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta x = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

is called the **average rate of change of** y **with respect to** x over the interval $[x_1, x_2]$.

The **instantaneous rate of change of** y **with respect to** x at $x = x_1$ is equal to

$$\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\substack{\cancel{\Delta x\to 0} \\ x_2\to x_1}} \frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x_1)$$