Section 2.8 **The derivative as a function**

A function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is called the **derivative of** f.

Example 1. Find the derivative of the function $f(x) = \dfrac{1-2x}{3+x}$ using the definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1-2(x+h)}{3+(x+h)} - \frac{1-2x}{3+x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(3+x)[1-2(x+h)] - (1-2x)(3+x+h)}{(3+x)(3+x+h)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{3+x-2x(3+x)-2h(3+x) - 3(1-2x)-x(1-2x)-h(1-2x)}{(3+x)(3+x+h)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\cancel{3}+\cancel{x}-\cancel{6x}-\cancel{2x^2}-6h-\cancel{2xh}-\cancel{3}+\cancel{6x}-\cancel{x}+\cancel{2x^2}-h+\cancel{2hx}}{(3+x)(3+x+h)}$$

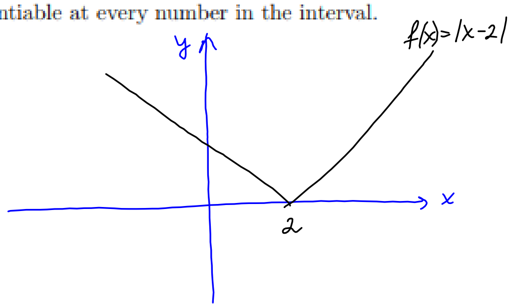
$$= \lim_{h\to 0} \frac{1}{\cancel{h}} \frac{-7\cancel{h}}{(3+x)(3+x+h)} = \lim_{h\to 0} \frac{-7}{(3+x)(3+x+h)} = \boxed{-\frac{7}{(3+x)^2}}$$

Definition. A function f is **differentiable at** a if $f'(a)$ exists.

It is **differentiable on an open interval** (a, b) if it is differentiable at every number in the interval.

Example 2. Where is the function $f(x) = |x - 2|$ differentiable?

$$|x-2| = \begin{cases} x-2, & \text{if } x-2 \geq 0 \\ -(x-2), & \text{if } x-2 < 0 \end{cases}$$



$f'(2)$

$$\lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{\substack{x \to 2 \\ x > 2}} \frac{x-2-0}{x-2} = 1$$

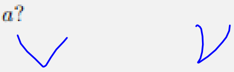
Don't match

$$\lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{\substack{x \to 2 \\ x < 2}} \frac{-(x-2)-0}{x-2} = -1$$

$f'(2)$ DNE

Theorem. If f is differentiable at a, then f is continuous at a

When is the function not differentiable at $x = a$?



- f has a "corner" or "kink" at a
- f is discontinuous at a
- the curve $y = f(x)$ has a vertical tangent line at $x = a$

Besides being able to find the function $f(x)$ from the function $f(x)$, you can also determine the graph of $f(x)$ from the graph of $f(x)$.

Things to look for when sketching $f(x)$ from the given graph of $f(x)$:

- Values of slopes of tangent lines to $f(x)$ (horizontal tangents = zeroes for f')
- Points where $f(x)$ has horizontal tangents
- Intervals where $f(x)$ is increasing or decreasing
- Intervals where $f(x)$ is concave up/down
- [redacted]

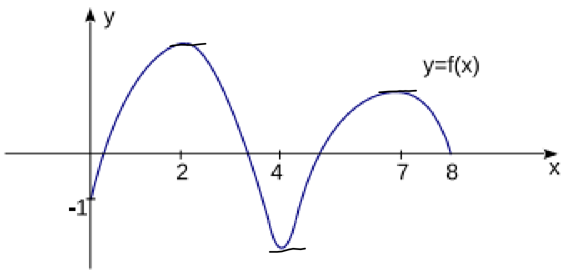
f is increasing $\Rightarrow f'(x) > 0$

f is decreasing $\Rightarrow f'(x) < 0$

f is concave up $\Rightarrow f'(x)$ is increasing

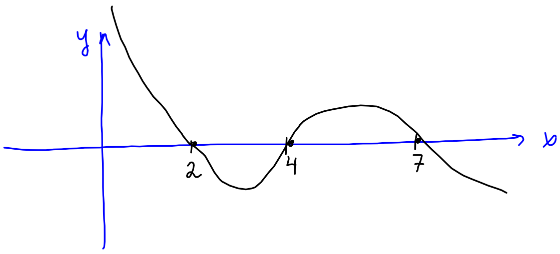
f is concave down $\Rightarrow f'(x)$ is decreasing

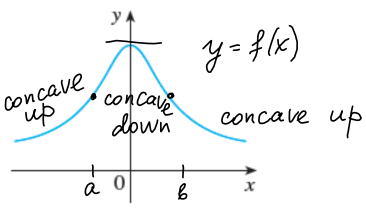
Example 3. Given the graph of $f(x)$ below, sketch the graph of $f'(x)$.



$f'(x) > 0$ on $(0,2), (4,7)$

$f'(x) < 0$ on $(2,4), (7,8)$



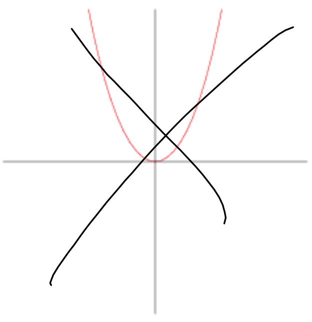


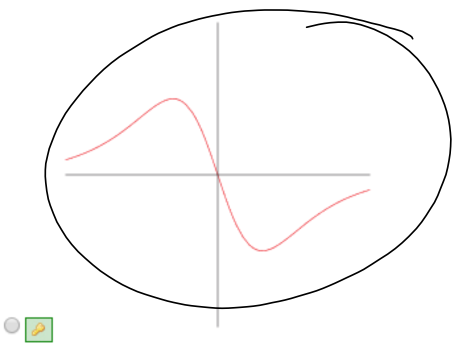
$f'(x) > 0$ on $(-\infty, 0)$

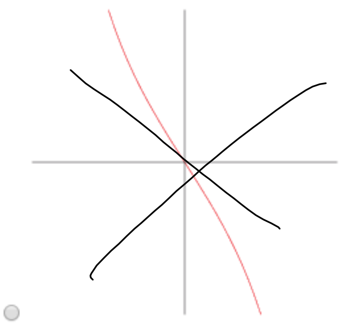
$f'(x) < 0$ on $(0, \infty)$

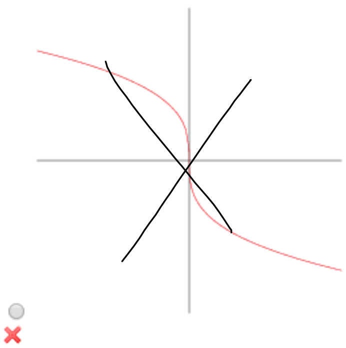
$f'(x)$ increasing on $(-\infty, a), (b, \infty)$

$f'(x)$ decreasing on (a, b)









Higher derivatives.

Leibnitz notations: $y'(x) = \frac{dy}{dx}$

Then

$$y''(x) = (y'(x))' = \frac{d^2y}{dx^2}$$ second-order derivative

$$y'''(x) = (y''(x))' = \frac{d^3y}{dx^3}$$ third-order derivative

$$y^{IV}(x) = y^{(4)}(x) = \frac{d^4y}{dx^4}$$ fourth-order derivative

$$y^{(n)}(x) = \frac{d^ny}{dx^n}$$ n-th order derivative

Example 4. If $f(x) = 2x^2 - x^3$, find f', f'', f''', $f^{(4)}$.

$$f'(x) = 4x - 3x^2$$
$$f''(x) = (4x - 3x^2)' = 4 - 6x$$
$$f'''(x) = (4 - 6x)' = -6$$
$$f^{IV}(x) = (-6)' = 0$$