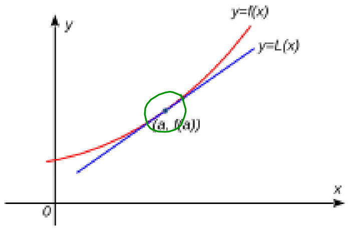
Section 3.10 **Linear approximations and differentials**

The equation of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$ can be used to approximate values of f that are near a.



tangent line

$y - f(a) = f'(a)(x-a)$

$y = f(a) + f'(a)(x-a)$

This is called the **linear approximation** or **tangent line approximation** of f at a

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

The function

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

is called the **linearization** of f at a.

Example 1. Find the linearization $L(x)$ of the function $f(x) = \dfrac{1}{\sqrt{2+x}}$ at $a = 0$

$L(x) = f(0) + f'(0)(x-0)$

$f(x) = (2+x)^{-1/2}$ | $f(0) = 2^{-1/2} = \frac{1}{\sqrt{2}}$

$f'(x) = -\frac{1}{2}(2+x)^{-3/2}$ | $f'(0) = -\frac{1}{2}\, 2^{-3/2} = -\frac{1}{2\cdot 2\sqrt{2}} = -\frac{1}{4\sqrt{2}}$

$$\boxed{L(x) = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}x}$$

Example 2. Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$.

$f(x) \approx f(0) + f'(0)(x-0)$

$f(x) = \sqrt{1-x}$ | $f(0) = \sqrt{1} = 1$

$f'(x) = \frac{1}{2}(1-x)^{-1/2}(-1)$ | $f'(0) = -\frac{1}{2}$

$$\boxed{\sqrt{1-x} \approx 1 - \frac{1}{2}x}$$

$\sqrt{0.9} = \sqrt{1-0.1} \approx 1 - \frac{1}{2}\cdot(0.1) = 1 - \frac{1}{20} = 0.95$

$\sqrt{0.99} = \sqrt{1-0.01} \approx 1 - \frac{1}{2}(0.01) = 1 - \frac{1}{200} = 0.995$

$\sqrt{0.9} \approx 0.948683\ldots$

$\sqrt{0.99} \approx 0.99498\,74$

Suppose the linear approximation for a function $f(x)$ at $a = 2$ is given by the tangent line $y = -2x + 11$.

What are $f(2)$ and $f'(2)$?

$f(2)$ = 7 ✗ 7

$f'(2)$ = -2 ✗ -2

If ~~$g(x) = [f(x)]^2$~~ $g(x) = e^{f(x)}$, find the linear approximation for $g(x)$ at $a = 2$.

$L(x)$ = ✗ $-28x + 105$

Linear approximation for $g(x)$ @ $x=2$

$g(x) \approx g(2) + g'(2)(x-2)$

$g(2) = e^{f(2)} = e^7$

$g'(x) = \left(e^{f(x)}\right)' = e^{f(x)} f'(x)$

$g'(2) = e^{f(2)} \cdot f'(2) = e^7(-2) = -2e^7$

$\boxed{g(x) \approx e^7 - 2e^7(x-2)} = -2e^7 \cdot x + 5e^7$

$f(x) \approx \underbrace{f(2) + f'(2)(x-2)}_{\text{"y" from the tangent line.}}$

$-2x + 11 = f(2) + f'(2)(x-2)$

$-2x + 11 = f(2) + f'(2)(x) - 2f'(2)$

Match the coefficients for the corresponding powers of x

x: $\boxed{-2 = f'(2)}$

1: $11 = f(2) - 2f'(2)$

$f(2) = 11 + 2f'(2)$

$= 11 - 4 = \boxed{7 = f(2)}$

10. Suppose the linear approximation for the function $f(x)$ at $a = 9$ is given by $y = 2x - 2$. If $g(x) = \sqrt{f(x)}$, find the linear approximation for $g(x)$ at $a = 9$.

(a) $L(x) = 4 + 2(x - 9)$

(b) $L(x) = 4 + \frac{1}{4}(x - 9)$

(c) $L(x) = 2 + 4(x - 9)$

(d) $L(x) = 4 + \frac{1}{2}(x - 9)$

(e) $L(x) = 2 - \frac{1}{2}(x - 9)$

$f(x) \approx \underbrace{f(9) + f'(9)(x-9)} = 2x - 2$

$\boxed{f'(9) = 2}$

$f(9) - 9f'(9) = -2$

$f(9) = -2 + 9f'(9) \quad \boxed{16 = f(9)}$

Linearization for $g(x)$ @ $x = 9$:

$L(x) = g(9) + g'(9)(x-9)$

$g(x) = \sqrt{f(x)}$

$g(9) = \sqrt{f(9)} = \sqrt{16} = 4$

$g'(x) = \frac{1}{2}[f(x)]^{-1/2} f'(x)$

$= \frac{f'(x)}{2\sqrt{f(x)}}$

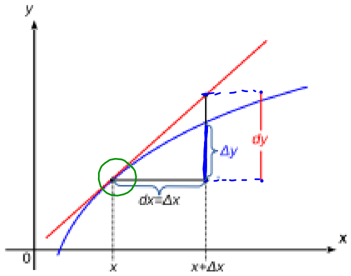
$g'(9) = \frac{f'(9)}{2\sqrt{f(9)}} = \frac{2}{8}$

$= \frac{1}{4}$

$L(x) = 4 + \frac{1}{4}(x-9) = \frac{1}{4}x + \frac{5}{4}$

Differentials.

We are using the notation $\dfrac{dy}{dx}$ to denote the derivative of y with respect to x, and it gives the slope of the tangent line at x. Now, we will actually view dy and dx as separate entities, called differentials.



$\dfrac{dy}{dx} = f'(x)$

$dy = f'(x)dx$

$\Delta y = f(x+\Delta x) - f(x)$

Definition Let $y = f(x)$, where f is a differentiable function. Then the **differential** dx is an independent variable; that is dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

$$\boxed{dy = f'(x)dx}$$

Example 3. Find dy if $y = x\tan x$.

$dy = (x\tan x)'dx$

$\boxed{dy = (\tan x + x\sec^2 x)dx}$

Example 4.

(a.) Find dy if $y = \sqrt{1-x}$

$dy = (\sqrt{1-x})'\,dx$

$dy = \frac{1}{2}(1-x)^{-1/2}(-1)dx$

$\boxed{dy = -\dfrac{1}{2\sqrt{1-x}}\,dx}$

(b.) Find the value of dy when $x = 0$ and $dx = .02$

$dy = -\dfrac{1}{2\sqrt{1-0}}(0.02) = -0.01$

$dy \approx \Delta y$, $\Delta y = f(a+\Delta x) - f(a)$, $dy = f'(a)dx$, $dx = \Delta x$

$f(a+\Delta x) \approx f(a) + f'(a)\Delta x$

Suppose that $f(a)$ is a known number and the approximate value is to be calculated for $f(a + \Delta x)$ where Δx is small. Then

$$\boxed{f(a + \Delta x) \approx f(a) + dy = f(a) + f'(a)\Delta x}$$

$\Delta y = f'(a)\Delta x$ is the error of approximation

Example 3. Use differentials to find an approximate value for

(a.) $\sqrt{36.1}$

$f(a+\Delta x) \approx f(a) + f'(a)\Delta x$

$a = 36$, $\Delta x = 0.1 = 36.1 - 36$

$f(x) = \sqrt{x}$	$f'(x) = \frac{1}{2\sqrt{x}}$
$f(36) = 6$	$f'(36) = \frac{1}{12}$

$\sqrt{36.1} = f(36 + 0.1) \approx f(36) + f'(36)(0.1)$

$= 6 + \frac{0.1}{12} = 6 + \frac{1}{120} = \frac{721}{120} \approx 6.008\bar{3}$

$\sqrt{36.1} \approx 6.00832755 43 \ldots$

(b.) $\sin 59°$

$f(x) = \sin x$

$a = 60° = \frac{\pi}{3}$ (rad)

$\Delta x = -1° = -\frac{\pi}{180}$ (rad)

convert degrees into radians.

$f(x) = \sin x$	$f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$
$f'(x) = \cos x$	$f'(\frac{\pi}{3}) = \frac{1}{2}$

$\sin 59° \approx f(\frac{\pi}{3}) + f'(\frac{\pi}{3})\Delta x$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{\pi}{180}\right) = \frac{180\sqrt{3} - \pi}{360} \approx 0.8573$

$\sin 59° \approx 0.8571673$

The radius of a circular disk is given as $\overbrace{17}^{r}$ cm with a maximum error in measurement of $\overbrace{0.2}^{\Delta r}$ cm.

(a) Use differentials to estimate the maximum error in the calculated area of the disk. (Round your answer to two decimal places.)

6.8π ✗ 21.36 cm^2

(b) What is the relative error? (Round your answer to four decimal places.)

0.0235 ✗ 0.0235

What is the percentage error? (Round your answer to two decimal places.)

2.35 ✗ 2.35 %

$A = \pi r^2$

error is $\Delta A = (\pi r^2)' \Delta r$

$= 2\pi r \Delta r$

$= 2\pi(17)(0.2) = 6.8\pi$

$\approx 21.36 \text{ cm}^2.$

relative error $\dfrac{\Delta A}{A} = \dfrac{6.8\cancel{\pi}}{\cancel{\pi}(17)^2} = \dfrac{0.4}{17}$

≈ 0.0235

percentage error $\dfrac{\Delta A}{A} \cdot 100\% = 2.35\%$

$$\frac{6.8\pi}{\pi(17)^2} = \frac{2\cancel{\pi}(\cancel{17})(0.2)}{\cancel{\pi}(17)^{\cancel{2}}} = \frac{0.4}{17}$$

$L(x) = f(a) + f'(a)(x-a)$ the linearization for $f(x)$ @ $x=a$.

$f(a+\Delta x) \approx f(a) + f'(a)\Delta x$

$a=2$, $f(x)=x^4$, $\Delta x = 0.01$

5. Use the linear approximation of $f(x) = x^4$ at $x = 2$ to find an approximate value of $(2.01)^4$.

(a) 16.12

(b) 16.26

(c) 16.32

(d) 16.48

(e) 16.80

$f(x) \approx f(2) + f'(2)(x-2)$

$f(x) = x^4$	$f'(x) = 4x^3$
$f(2) = 16$	$f'(2) = 32$

$f(x) \approx 16 + 32(x-2)$

$f(2.01) \approx 16 + 32(2.01-2)$

$= 16 + 32(0.01) = 16.32$

10. Suppose the linear approximation for the function $f(x)$ at $a = 9$ is given by $y = 2x - 2$. If $g(x) = \sqrt{f(x)}$, find the linear approximation for $g(x)$ at $a = 9$.

(a) $L(x) = 4 + 2(x - 9)$

(b) $L(x) = 4 + \frac{1}{4}(x - 9)$

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(d) $L(x) = 4 + \frac{1}{2}(x - 9)$

(e) $L(x) = 2 - \frac{1}{2}(x - 9)$

$L(x) = f(a) + f'(a)(x-a)$ linearization

$f(x) = ?$, $a = 9$.

$L(x) = 2x - 2$ find $f'(9)$ and $f(9)$.

$f(9) + f'(9)(x-9) = 2x - 2$

$f'(9)(x) + [f(9) - 9f'(9)] = 2x - 2$

x: $\boxed{f'(9) = 2}$

1: $f(9) - 9f'(9) = -2$

$f(9) = -2 + 9f'(9) = -2 + 18 = \boxed{16 = f(9)}$

$g(x) = \sqrt{f(x)}$

$L(x) = g(9) + g'(9)(x-9)$

$g(x) = \sqrt{f(x)}$

$g(9) = \sqrt{f(9)} = \sqrt{16} = 4$

$g'(x) = \frac{1}{2}[f(x)]^{-1/2} f'(x)$

$g'(9) = \frac{1}{2}[f(9)]^{-1/2} f'(9)$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{16}} \cdot (2) = \frac{1}{4}$

4. Find $f^{(2018)}(x)$ for $f(x) = xe^{-x}$.

(a) xe^{-x}

(b) $2018xe^{-x}$

(c) xe^{-2018x}

(d) $(2018 - x)e^{-x}$

(e) $(x - 2018)e^{-x}$

$f(x) = xe^{-x}$

$f'(x) = e^{-x} + xe^{-x}(-1)$

$= e^{-x} - xe^{-x}$

$f''(x) = -e^{-x} - e^{-x} + xe^{-x}$

$= -2e^{-x} + xe^{-x}$

$f'''(x) = +2e^{-x} + e^{-x} - xe^{-x}$

$= 3e^{-x} - xe^{-x}$

$f^{(2018)}(x) = -2018e^{-x} + xe^{-x}$

$f^{(151)}(x) = 151e^{-x} - xe^{-x}$