Section 3.4 **The Chain Rule**

> If the derivatives $g'(x)$ and $f'(g(x))$ both exist, and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F'(x)$ exists an is given by the product
>
> $$\boxed{[f\,(g(x))]' = F'(x) = f'(g(x))g'(x)}$$

Example 1. If $f(x) = x^2 + 4x - 3$ and $g(x) = \sin x$, write equations for $f\,(g(x))$ and $g\,(f(x))$.

$f(g(x)) = f(\sin x) = \sin^2 x + 4 \sin x - 3$

$g(f(x)) = \sin(x^2 + 4x - 3)$

Example 2. Suppose that $F(x) = f(g(x))$, where $g(2) = 5$, $g'(2) = 4$, $f(2) = 3$, $f'(2) = -2$, and $f'(5) = 11$. Find $F'(2)$.

$F'(2) = f'(\overset{5}{g(2)})\,\overset{4}{g'(2)} = 4\,\overset{11}{f'(5)} = 4(11) = 44$

If n is any real number and $u = g(x)$ is differentiable, then

$$\boxed{\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)}$$

If n is any real number and $u = g(x)$ is differentiable, then

$$\boxed{\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}}$$

Example 3. Find the derivative of each function

1. $y = \sec(2x)$

$$y' = \sec(2x)\tan(2x)\,(2x)'$$
$$= 2\sec(2x)\tan(2x)$$

2. $y = \sin(x^2)$

$$y' = \cos(x^2)\,(x^2)'$$
$$= 2x\cos(x^2)$$

3. $y = (1 + \cos^2 x)^6$

$$y' = 6(1+\cos^2 x)^5\,(1+\cos^2 x)'$$
$$= 6(1+\cos^2 x)^5\; 2\cos x\,(\cos x)'$$
$$= 6(1+\cos^2 x)^5\; 2(\cos x)(-\sin x)$$

4. $y = (1 + \sqrt{x^2 + 2})^3$

$$y' = 3\left(1+\sqrt{x^2+2}\right)^2 \Big(1+\overbrace{\sqrt{x^2+2}}^{(x^2+2)^{1/2}}\Big)'$$
$$= 3\left(1+\sqrt{x^2+2}\right)^2 \tfrac{1}{2}(x^2+2)^{-1/2}\,(x^2+2)'$$
$$= \boxed{3\left(1+\sqrt{x^2+2}\right)^2 \tfrac{1}{2}(x^2+2)^{-1/2}\,(2x)}$$

5. $y=\sqrt[4]{\frac{t^3+1}{t^3-1}} = \left(\frac{t^3+1}{t^3-1}\right)^{1/4}$

$$y' = \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{1/4-1}\left(\frac{t^3+1}{t^3-1}\right)' = \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4}\frac{(t^3+1)'(t^3-1)-(t^3-1)'(t^3+1)}{(t^3-1)^2}$$

$$= \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4}\frac{3t^2(t^3-1)-3t^2(t^3+1)}{(t^3-1)^2} = \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4}\frac{\cancel{3t^5}-3t^2-\cancel{3t^5}-3t^2}{(t^3-1)^2}$$

$$= \boxed{\frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4}\frac{-6t^2}{(t^3-1)^2}}$$

$$= -\frac{3}{2}t^2\left(\frac{t^3-1}{t^3+1}\right)^{3/4}\frac{1}{(t^3-1)^2}$$

$$= \boxed{-\frac{3}{2}t^2\frac{1}{(t^3+1)^{3/4}(t^3-1)^{5/4}}}$$

6. $y=\sin^2(\cos 4x)$

$$y' = 2\sin(\cos 4x)(\cos 4x)'$$

$$= 2\sin(\cos 4x)\cdot(-\sin 4x)(4x)'$$

$$= 2\sin(\cos 4x)\cdot(-\sin 4x)(4)$$

$$= \boxed{-8\sin 4x\,\sin(\cos 4x)}$$

7. $y=e^{x\sin 2x}$

$$y' = e^{x\sin 2x}(x\sin 2x)'$$

$$= e^{x\sin 2x}\left((x)'\sin 2x + x(\sin 2x)'\right)$$

$$= e^{x\sin 2x}\left(\sin 2x + x\cos 2x\,(2x)'\right)$$

$$= \boxed{e^{x\sin 2x}\left(\sin 2x + x\cos 2x\,(2)\right)}$$

Example 4. Find the equation of the tangent line to the curve $y = \dfrac{8}{\sqrt{4+3x}}$ at the point (4,2).

an equation of the tangent line to $y = f(x)$
@ the point where $x = a$ is

$$\boxed{y - f(a) = f'(a)(x-a)}$$

$a = 4$, $f(a) = 2$

$$y = 8(4+3x)^{-1/2}$$

$$y' = 8 \cdot \left(-\tfrac{1}{2}\right)(4+3x)^{-1/2-1}(4+3x)'$$

$$= -4(4+3x)^{-3/2}(3) = \frac{-12}{(4+3x)^{3/2}}$$

$$y'(4) = \frac{-12}{(4+12)^{3/2}} = -\frac{12}{64} = -\frac{3}{16}$$

Tangent line: $\boxed{y - 2 = -\frac{3}{16}(x-4)}$