Section 3.5 **Implicit differentiation.**

Some functions are defined implicitly by a relation between x and y, where x is the independent variable and y depends on x. In order to find the derivative of y with respect to x, we can use the method of **implicit differentiation**. This consists of differentiating both sides of the relation with respect to x and then solving the resulting equation for y'.

$y = y(x)$, $\frac{dy}{dx}$, $(x') = 1$

Example 1. Find dy/dx by implicit differentiation.

1. $(x^2 - xy + y^3)' = (8)'$

$2x - (xy)' + (y^3)' = 0$

$2x - [(x)'y + xy'] + 3y^2 y' = 0$

$2x - y - \underbrace{xy' + 3y^2 y'} = 0$ solve for y'

$y'(3y^2 - x) = y - 2x$

$\boxed{y' = \frac{y-2x}{3y^2-x}}$

2. $(xe^y)' = (x-y)'$

$x'e^y + x(e^y)' = x' - y'$

$e^y + xe^y y' = 1 - y'$

$xe^y y' + y' = 1 - e^y$

$y'(xe^y+1) = 1 - e^y$

$$\boxed{y' = \frac{1-e^y}{xe^y+1}}$$

3. $\left(\tan(x-y)\right)' = \left(\dfrac{y}{1+x^2}\right)'$

$$\sec^2(x-y)\,(x-y)' = \frac{y'(1+x^2)-(1+x^2)'y}{(1+x^2)^2}$$

$$\sec^2(x-y)\,(1-y') = \frac{y'(1+x^2)-2xy}{(1+x^2)^2}$$

$$\sec^2(x-y) - y'\sec^2(x-y) = \frac{y'\cancel{(1+x^2)}}{(1+x^2)^{\cancel{2}}} - \frac{2xy}{(1+x^2)^2}$$

$$\sec^2(x-y) + \frac{2xy}{(1+x^2)^2} = \frac{y'}{1+x^2} + y'\sec^2(x-y)$$

$$\sec^2(x-y) + \frac{2xy}{(1+x^2)^2} = y'\left[\frac{1}{1+x^2} + \sec^2(x-y)\right]$$

$$y' = \frac{\sec^2(x-y) + \frac{2xy}{(1+x^2)^2}}{\frac{1}{1+x^2} + \sec^2(x-y)} = \frac{\frac{(1+x^2)^2\sec^2(x-y)+2xy}{(1+x^2)^2}}{\frac{1+(1+x^2)\sec^2(x-y)}{1+x^2}}$$

$$y' = \frac{(1+x^2)^2\sec^2(x-y)+2xy}{(1+x^2)^{\cancel{2}}} \cdot \frac{\cancel{1+x^2}}{1+(1+x^2)\sec^2(x-y)}$$

$$\boxed{y' = \frac{(1+x^2)^2\sec^2(x-y)+2xy}{(1+x^2)+(1+x^2)^2\sec^2(x-y)}}$$

$x = x(y)$, $y' = 1$

Example 2. Let y be the independent variable and x be the dependent variable. Use implicit differentiation to find dx/dy if

$\frac{dx}{dy}$

$$\left((x^2+y^2)^2\right)' = \left(4x^2y\right)' \quad \text{for } y.$$

$$2(x^2+y^2)\frac{d}{dy}(x^2+y^2) = 4\left[\frac{d}{dy}(x^2)y + x^2\cancel{\frac{d}{dy}}y'\right]$$

$$\cancel{2}(x^2+y^2)\left(\cancel{2}x\frac{dx}{dy} + \cancel{2}y\right) = \cancel{4}y(2x)\frac{dx}{dy} + \cancel{4}x^2$$

solve for $\frac{dx}{dy}$

$$(x^2+y^2)\left(\frac{dx}{dy} + y\right) = 2xy\frac{dx}{dy} + x^2$$

$$(x^2+y^2)\frac{dx}{dy} + (x^2+y^2)y = 2xy\frac{dx}{dy} + x^2$$

$$(x^2+y^2)\frac{dx}{dy} - 2xy\frac{dx}{dy} = x^2 - (x^2+y^2)y$$

$$\frac{dx}{dy}\left[x^2+y^2-2xy\right] = x^2-(x^2+y^2)y$$

$$\boxed{\frac{dx}{dy} = \frac{x^2-(x^2+y^2)y}{x^2+y^2-2xy}}$$

Derivatives of the inverse trigonometric functions.

Let us find $(\arcsin x)'$: $y = \arcsin x \Rightarrow \frac{d}{dx}x = \frac{d}{dx}\sin y$, note that $y = y(x)$

$1 = \cos y \cdot y'$

$\cos^2 y + \sin^2 y = 1$ or $\cos y = \sqrt{1-\sin^2 y}$

$y' = \frac{1}{\cos y}$

$\sin y = x$

$y' = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

$$\boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}}$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

Example 3. Find the derivative.

1. $f(x) = \arccos(x^2)$

$$f'(x) = (\arccos(x^2))' = -\frac{1}{\sqrt{1-(x^2)^2}}(x^2)' = \boxed{-\frac{2x}{\sqrt{1-x^4}}}$$

2. $f(x) = \arctan\sqrt{\frac{1-x}{1+x}}$ $\qquad (\arctan x)' = \frac{1}{1+x^2}$

$$f'(x) = \frac{1}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2}\left(\sqrt{\frac{1-x}{1+x}}\right)' = \frac{1}{1+\frac{1-x}{1+x}}\cdot\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\left(\frac{1-x}{1+x}\right)'$$

$$= \frac{1}{\frac{1+x+(1-x)}{1+x}}\cdot\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\frac{(1-x)'(1+x)-(1+x)'(1-x)}{(1+x)^2}$$

$$= \underbrace{\frac{\cancel{1+x}}{2}\cdot\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\frac{-1(x+1)-(1-x)}{(1+x)^{\cancel{2}}}}$$ $= -x-1-1+x = -2$

$$= \frac{1}{2\cancel{4}}\left(\frac{1-x}{1+x}\right)^{-1/2}\cdot\frac{-\cancel{2}}{1+x} = -\frac{1}{2}\left(\frac{\cancel{1+x}}{1-x}\right)^{1/2}\cdot\frac{1}{[(1+x)^{1/2}]^{\cancel{2}}}$$

$-\frac{1}{2}\frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{1}{(\sqrt{1+x})^2}$

$$= -\frac{1}{2}\frac{1}{(1-x)^{1/2}(1+x)^{1/2}} = -\frac{1}{2}\frac{1}{[(1-x)(1+x)]^{1/2}}$$

$$= \boxed{-\frac{1}{2}\frac{1}{\sqrt{1-x^2}}}$$