Section 3.6 **Derivatives of logarithmic functions**

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}} \qquad \boxed{\frac{d}{dx}\log_a x = \frac{1}{x\ln a}}$$

$$\boxed{\frac{d}{dx}\ln g(x) = \frac{g'(x)}{g(x)}} = \frac{1}{g(x)}\, g'(x)$$

Example 1. Differentiate each function

1. $f(x) = \ln(\sin^2 x)$

$$f'(x) = \frac{1}{\sin^2 x}\left(\sin^2 x\right)' = \frac{1}{\sin^{\cancel{2}} x}\, 2\cancel{\sin x}\,\overset{\cos x}{(\cancel{\sin x})'}$$

$$= \frac{2\cos x}{\sin x} = \boxed{2\cot x}$$

2. $\frac{d}{dx}\log_3(\tan x^2)$

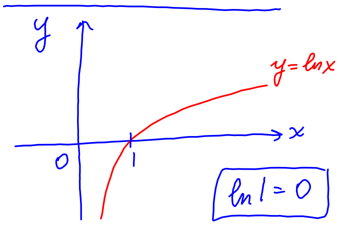
$$= \frac{1}{\tan(x^2)\ln 3}\left(\tan x^2\right)'$$

$$= \frac{1}{\tan(x^2)\ln 3}\sec^2(x^2)\,(x^2)'$$

$$= \boxed{\frac{1}{\tan(x^2)\ln 3}\sec^2(x^2)\,(2x)}$$

$y = \ln x \Leftrightarrow x = e^y$

$y = \log_a x \Leftrightarrow x = a^y$, $a > 0,\ a \neq 1$



$\boxed{\ln 1 = 0}$

$\lim_{x\to 0^+} \ln x = -\infty$

$\lim_{x\to \infty} \ln x = \infty$

3. $f(x) = \ln|x|$

$$\ln|x| = \begin{cases} \ln x, & \text{if } x > 0 \\ \ln(-x), & \text{if } x < 0 \end{cases}$$

$$(\ln|x|)' = \begin{cases} (\ln x)', & \text{if } x > 0 \\ (\ln(-x))', & \text{if } x < 0 \end{cases} = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \frac{1}{-x}(-x)', & \text{if } x < 0 \end{cases} = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \frac{-1}{-x}, & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \frac{1}{x}, & \text{if } x < 0 \end{cases}$$

$$\boxed{(\ln|x|)' = \frac{1}{x},\quad x \neq 0}$$

4. $f(x) = \sqrt{\frac{9 - x^2}{9 + x^2}}$

logarithmic differentiation

$$\ln f = \ln\sqrt{\frac{9-x^2}{9+x^2}}$$

$$\ln f = \frac{1}{2}\ln\frac{9-x^2}{9+x^2}$$

$$\ln f = \frac{1}{2}\ln\frac{(3-x)(3+x)}{9+x^2}$$

$$\boxed{\begin{aligned} &\ln(xy) = \ln x + \ln y \\ &\ln\frac{x}{y} = \ln x - \ln y \end{aligned}}$$

$$\ln f = \frac{1}{2}\left[\ln(3-x) + \ln(3+x) - \ln(9+x^2)\right]$$

Differentiate for x:

$$\frac{f'}{f} = \frac{1}{2}\left[\frac{1}{3-x}(3-x)' + \frac{1}{3+x} - \frac{1}{9+x^2}(9+x^2)'\right]$$

$$\frac{f'}{f} = \frac{1}{2}\left[\frac{-1}{3-x} + \frac{1}{3+x} - \frac{2x}{9+x^2}\right]$$

solve for f':

$$f' = f\cdot\frac{1}{2}\left[\frac{-1}{3-x} + \frac{1}{3+x} - \frac{2x}{9+x^2}\right]$$

$$\boxed{f' = \frac{1}{2}\sqrt{\frac{9-x^2}{9+x^2}}\left(\frac{-1}{3-x} + \frac{1}{3+x} - \frac{2x}{9+x^2}\right)}$$

Example 2. Find y' and y'' if

$$y = \sqrt{x}\ln x.$$

$$y' = (\sqrt{x}\ln x)' = (\sqrt{x})'\ln x + \sqrt{x}(\ln x)'$$
$$= \frac{1}{2}x^{-1/2}\ln x + \sqrt{x}\cdot\frac{1}{x}$$
$$y' = \frac{1}{2}\frac{\ln x}{\sqrt{x}} + \frac{1}{\sqrt{x}} = \frac{2+\ln x}{2\sqrt{x}}$$

$$\frac{\sqrt{x}}{x} = \frac{\cancel{\sqrt{x}}}{(\sqrt{x})^{\cancel{2}}}$$

$$y'' = \left(\frac{2+\ln x}{2\sqrt{x}}\right)' = \frac{(2+\ln x)'2\sqrt{x} - (2\sqrt{x})'(2+\ln x)}{(2\sqrt{x})^2}$$

$$= \frac{\frac{1}{x}2\sqrt{x} - \cancel{2}\frac{1}{\cancel{2}}x^{-1/2}(2+\ln x)}{4x} = \frac{\frac{2}{\sqrt{x}} - \frac{2+\ln x}{\sqrt{x}}}{4x} = \boxed{\frac{-\ln x}{4x\sqrt{x}}}$$

$$(e^x)' = e^x$$

$$\boxed{a^x = e^{x\ln a}} \qquad (e^{x\ln a})' = e^{x\ln a}\cdot\ln a = a^x\ln a$$

$$\boxed{(a^x)' = a^x\ln a}, \quad a > 0, a \neq 1.$$

Example 3. Find $\frac{d}{dx}\left(\sqrt{2-3^x} + \pi^{-x} + x^e\right)$

$$= \frac{1}{2}(2-3^x)^{-1/2}(2-3^x)' + (\pi^{-x})' + (x^e)'$$
$$= \frac{1}{2}(2-3^x)^{-1/2}(-3^x\ln 3) + \pi^{-x}\ln\pi\,(\cancel{-x})'^{\,-1} + e x^{e-1}$$
$$= \boxed{-\frac{1}{2}(2-3^x)^{-1/2}\,3^x\ln 3 - \pi^{-x}\ln\pi + e x^{e-1}}$$

Logarithmic differentiation

> Steps in logarithmic differentiation
>
> 1. Take the logarithm of both sides of an equation.
> 2. Differentiate implicitly with respect to x.
> 3. Solve the resulting equation for y'.

$$\boxed{[f(x)]^{g(x)} = e^{g(x)\ln[f(x)]}}$$

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Example 4. Differentiate each function

$\boxed{\ln(x^n) = n\ln x}$

1. $y = x^x$

$\ln y = \ln(x^x)$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}\, x\ln x$

implicit differentiation

$\frac{1}{y}y' = (x)'\ln x + x(\ln x)'$

$\frac{1}{y}y' = \ln x + x\cdot\frac{1}{x}$

$\frac{1}{y}y' = 1 + \ln x$

solve for y':

$y' = y(1+\ln x)$

$\boxed{y' = x^x(1+\ln x)}$

2. $y = x^{\sin x}$

$\ln y = \ln(x^{\sin x})$

$\ln y = \sin x \ln x$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}\sin x\ln x$

$\frac{1}{y}y' = (\sin x)'\ln x + (\sin x)(\ln x)'$

$\frac{1}{y}y' = \cos x\ln x + \sin x\,\frac{1}{x}$

$y' = y\left(\cos x\ln x + \frac{\sin x}{x}\right)$

$\boxed{y' = x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right)}$

3. $y = \cos(x^{\sqrt{x}})$

$\boxed{a^x = e^{x\ln a}}$

$y' = -\sin(x^{\sqrt{x}})\cdot(x^{\sqrt{x}})'$

$= -\sin(x^{\sqrt{x}})\cdot(e^{\sqrt{x}\ln x})'$

$= -\sin(x^{\sqrt{x}})\cdot e^{\sqrt{x}\ln x}\underbrace{(\sqrt{x}\ln x)'}_{\text{product rule}}$

$= -\sin(x^{\sqrt{x}})\, e^{\sqrt{x}\ln x}\left(\frac{1}{2}x^{-1/2}\ln x + \cancel{\sqrt{x}}\cdot\frac{1}{\cancel{x^{1/2}}}\right)$

$= \boxed{-\sin(x^{\sqrt{x}})\, x^{\sqrt{x}}\left(\frac{1}{2}x^{-1/2}\ln x + \frac{1}{\sqrt{x}}\right)}$

4. $\ln y = \ln\frac{(x+1)^4\sqrt[5]{x^2+1}}{(x^3-1)^{151}(1+3x^2)^{2018}}$

$\ln y = \ln(x+1)^4 + \ln\sqrt[5]{x^2+1} - \left[\ln(x^3-1)^{151} + \ln(1+3x^2)^{2018}\right]$

$\ln y = 4\ln(x+1) + \frac{1}{5}\ln(x^2+1) - 151\ln(x^3-1) - 2018\ln(1+3x^2)$

Differentiate for x.

$\frac{1}{y}y' = 4\cdot\frac{1}{x+1} + \frac{1}{5}\,\frac{1}{x^2+1}(x^2+1)' - 151\,\frac{1}{x^3-1}(x^3-1)' - 2018\,\frac{1}{1+3x^2}(1+3x^2)$

$\frac{1}{y}y' = \frac{4}{x+1} + \frac{2x}{5(x^2+1)} - \frac{151(3x^2)}{x^3-1} - \frac{2018(6x)}{1+3x^2}$

$y' = y\left(\frac{4}{x+1} + \frac{2x}{5(x^2+1)} - \frac{453x^2}{x^3-1} - \frac{12548x}{1+3x^2}\right)$

$$\boxed{y' = \frac{(x+1)^4\sqrt[5]{x^2+1}}{(x^3-1)^{151}(1+3x^2)^{2018}}\left(\frac{4}{x+1} + \frac{2x}{5(x^2+1)} - \frac{453x^2}{x^3-1} - \frac{12548x}{1+3x^2}\right)}$$

Example 5. Find y' if $x^y = y^x$.

$$\ln(x^y) = \ln(y^x)$$

$$\frac{d}{dx} y \ln x = \frac{d}{dx} x \ln y$$

differentiate both parts for x,
assume that $y = y(x)$

$$y' \ln x + y (\ln x)' = x' \ln y + x (\ln y)'$$

$$y' \ln x + y \cdot \frac{1}{x} = \ln y + x \cdot \frac{1}{y} \cdot y'$$

$$y' \ln x - \frac{x}{y} y' = \ln y - \frac{y}{x}$$

solve for y':

$$y' \left(\ln x - \frac{x}{y} \right) = \ln y - \frac{y}{x}$$

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - x/y} = \frac{\frac{x \ln y - y}{x}}{\frac{y \ln x - x}{y}}$$

$$= \frac{x \ln y - y}{x} \cdot \frac{y}{y \ln x - x} = \boxed{\frac{y(x \ln y - y)}{x(y \ln x - x)}}$$