Section 3.8 **Exponential growth and decay**

If $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to $y(t)$ at any time, then $\frac{dy}{dt}$

$$\frac{dy}{dt} = ky$$

where k is a constant. This equation is called the law of natural growth if $k > 0$ or the the law of natural decay if $k < 0$.
The only solution to this equation is

$$\boxed{y(t) = y(0)e^{kt}}$$

Example 1. A bacteria culture starts with 500 bacteria and after 3 hours there are 8000 bacteria.

1. Find an expression for the number of bacteria after t hours.

$y(t)$ is the population of bacteria at time t, $t \geq 0$
t is in hours.

$y(0) = 500$, $y(3) = 8000$

$y(t) = y(0)e^{kt}$

$y(t) = 500e^{kt}$, k is unknown const.

$y(3) = 500e^{3k} = 8000$ solve for k.

$e^{3k} = \frac{80}{5} = 16$, $\ln e^{3k} = \ln 16 \Rightarrow 3k = \ln 16$, $k = \frac{\ln \overbrace{16}^{2^4}}{3} = \frac{4\ln 2}{3}$

$y(t) = 500e^{\frac{4\ln 2}{3}t} = 500\left(e^{\ln 2}\right)^{4t/3} = \boxed{500 \cdot 2^{\frac{4t}{3}} = y(t)}$

2. Find the number of bacteria after 4 hours.

$$\boxed{y(4) = 500 \cdot 2^{\frac{16}{3}}}$$

$$\boxed{\log_a t = \frac{\ln t}{\ln a}}$$

3. When will the population reach 30,000?

Find t such that $y(t) = 30{,}000$

$500 \cdot 2^{\frac{4t}{3}} = 30{,}000$

$\log_2 2^{\frac{4t}{3}} = \frac{300}{5} = \log_2 60 \Rightarrow \frac{4t}{3} = \log_2(60)$

$\boxed{t = \frac{3}{4}\log_2(60)} = \frac{3}{4}\frac{\ln 60}{\ln 2}$

Example 2. Polonium-214 has a half-life of 1.4×10^{-4} s.

1. If a sample has a mass of 50 mg, find a formula for the mass that remains after t seconds.

$m(t)$ is the mass of the sample @ time t.

$m(t) = m(0)e^{kt}$

$m(0) = 50$

$m(t) = 50e^{kt}$

$\underbrace{m(1.4\times10^{-4})} = \frac{m(0)}{2} = 25$

$50 \cdot e^{k \cdot 1.4\times10^{-4}} = 25$

$\ln e^{k\cdot 1.4\times10^{-4}} = \ln\frac{1}{2}$

$k \cdot 1.4\times10^{-4} = \ln\frac{1}{2} = -\ln 2$

$k = -\frac{\ln 2}{1.4\times10^{-4}} = -\frac{\ln 2}{1.4}\times10^{4}$

$m(t) = 50\cdot e^{-\frac{\ln 2}{1.4}\times10^4 t}$

$= 50\cdot\left(e^{\ln 2}\right)^{\frac{-10^4}{1.4}t}$

$\boxed{m(t) = 50\cdot 2^{-\frac{10^4}{1.4}t}}$

2. Find the mass that remains after a hundredth of a second.

$m(0.01) = 50\cdot 2^{\frac{-10^4}{1.4}\cdot 10^{-2}} = \boxed{50\cdot 2^{-\frac{100}{1.4}}}$

3. How long would it take for the mass to decay to 40 mg?

Find t such that $m(t) = 40$

$\frac{50\cdot 2^{-\frac{10^4}{1.4}t}}{50} = \frac{40}{50}$ solve for t.

$\log_2 2^{-\frac{10^4}{1.4}t} = \log_2\frac{4}{5}$

$\frac{-10^4}{1.4}t = \log_2\frac{4}{5}$

$\boxed{t = -\frac{1.4}{10^4}\log_2\frac{4}{5}}$

Newtons Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings.

$$\frac{dy}{dt} = k(y - M)$$

where y is the temperature of the object and M the temperature of the surroundings. The solution of this equation is a function of the form:

$$\boxed{y(t) = (y_0 - M)e^{kt} + M}$$

where y_0 is the initial temperature of the object, $y_0 = y(0)$.

Example 3. A roast turkey is taken from the oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

1. If the temperature of turkey is 150°F after half an hour, what is the temperature after 45 min?

$T(t)$ is the temperature of the turkey @ time t (t in min)

$T(t) = (T(0) - M)e^{kt} + M$

$T(0) = 185, \quad M = 75$

$T(t) = (185 - 75)e^{kt} + 75$

$T(t) = 110e^{kt} + 75 \implies T(30) = 110e^{30k} + 75 = 150$, solve for k.

$T(30) = 150$

$110e^{30k} = 75$

$\ln e^{30k} = \frac{75}{110} = \ln\frac{15}{22}$

$30k = \ln\frac{15}{22}$

$k = \frac{1}{30}\ln\frac{15}{22}$

$T(t) = 110e^{\frac{t}{30}\ln\frac{15}{22}} + 75$

$T(45) = 110e^{\frac{45}{30}\ln\frac{15}{22}} + 75 = \boxed{110e^{\frac{3}{2}\ln\frac{15}{22}} + 75} = 110\left(e^{\ln\frac{15}{22}}\right)^{3/2} + 75$

$= \boxed{110 \cdot \left(\frac{15}{22}\right)^{3/2} + 75}$

2. When will the turkey have cooled to 100°F ? $T(t) = 100$

Find t such that

$110e^{\frac{t}{30}\ln\frac{15}{22}} + 75 = 100$

$\frac{110e^{\frac{t}{30}\ln\frac{15}{22}}}{110} = \frac{25}{110}$

$\ln e^{\frac{t}{30}\ln\frac{15}{22}} = \frac{25}{110} = \ln\frac{5}{22}$

$\frac{t}{30}\ln\frac{15}{22} = \ln\frac{5}{22}$

$\frac{t}{30} = \frac{\ln\frac{5}{22}}{\ln\frac{15}{22}}$

$\boxed{t = 30\frac{\ln\frac{5}{22}}{\ln\frac{15}{22}}}$

$\ln\frac{a}{b} = \ln a - \ln b$