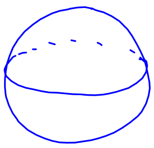
Quiz tomorrow: derivatives of vector functions
slopes and tangents of parametric curves, section 3.7

Section 3.9 **Related rates**

Strategy

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 1. A spherical snowball is melting in such a way that its volume is decreasing at a rate of 1 cm^3/min. At what rate is the diameter decreasing when the diameter is 10 cm.



D is the diameter of the snowball
V is the volume of the snowball

$\frac{dD}{dt}$ rate of change of the diameter
$\frac{dV}{dt}$ rate of change of the volume

$$\frac{dV}{dt} = -1, \quad D = 10, \quad \frac{dD}{dt} = ?$$

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{\pi}{6}D^3, \qquad V = V(t), \ D = D(t)$$

$$\frac{d}{dt}V = \frac{d}{dt}\frac{\pi}{6}D^3$$

$$\frac{dV}{dt} = \frac{\pi}{6}\cdot 3D^2\frac{dD}{dt} \Rightarrow \frac{dD}{dt} = \frac{2}{\pi D^2}\frac{dV}{dt} = \frac{2}{\pi(10)^2}(-1) = \boxed{-\frac{1}{50\pi}}$$

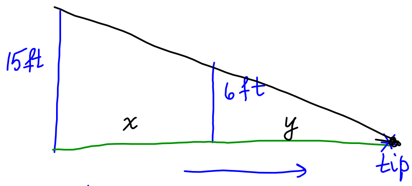
Example 2. A street light is at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path.

(a.) How fast is the tip of his shadow moving when he is 40 ft from the pole?

x is the distance from the man to the pole

y is the length of his shadow

$\frac{dx}{dt} = 5$ ft/s (speed of the man), $x = 40$.



$\frac{d(x+y)}{dt} = ?$

similar right triangles.

$\frac{x+y}{15} = \frac{y}{6}$

solve for y:

$6(x+y) = 15y$

$6x + 6y = 15y$

$9y = 6x$

$y = \frac{6x}{9} = \frac{2x}{3}$

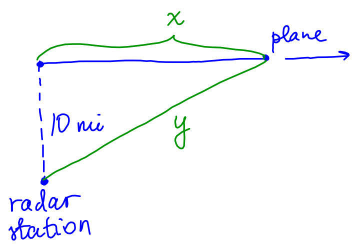
$$\frac{d(x+y)}{dt} = \frac{d(x+\frac{2}{3}x)}{dt} = \frac{d(\frac{5}{3}x)}{dt} = \frac{5}{3}\frac{dx}{dt} = \boxed{\frac{25}{3}\text{(ft/s)}}$$

(b.) How fast is his shadow lengthening at that point?

$\frac{dy}{dt} = ?$

$$\frac{dy}{dt} = \frac{d(\frac{2}{3}x)}{dt} = \frac{2}{3}\frac{dx}{dt} = \frac{2}{3}(5) = \boxed{\frac{10}{3}\text{(ft/s)}}$$

Example 3. A plane flying horizontally at an altitude of 10 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 20 mi away from the station.



y is the distance from the plane to the station.

$$\frac{dx}{dt} = 500$$

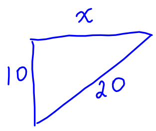
$$\frac{dy}{dt} = ? \quad \text{when} \quad y = 20\text{ mi}$$

$$\frac{d}{dt}\left(y^2\right) = \frac{d}{dt}\left(x^2 + 100\right)$$

$$\frac{\cancel{2}y\,\frac{dy}{dt}}{y} = \frac{\cancel{2}x\,\frac{dx}{dt}}{y} + 0$$

$$\frac{dy}{dt} = \frac{x}{y}\,\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{10\sqrt{3}}{20}(500) = \boxed{250\sqrt{3}}\left(\text{mi}/\text{h}\right)$$

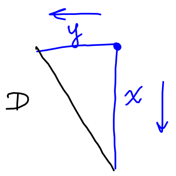


$$x^2 = 20^2 - 10^2$$

$$x^2 = 300$$

$$x = \sqrt{300} = 10\sqrt{3}$$

Example 4. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?



x is the distance traveled by the first car

y is the distance traveled by the 2nd car

$\frac{dx}{dt} = 60, \quad \frac{dy}{dt} = 25$

D is the distance between the cars.

$\frac{dD}{dt} = ?$

$\frac{d}{dt} D^2 = \frac{d}{dt}(x^2 + y^2)$

$\cancel{2}D \frac{dD}{dt} = \cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt}$

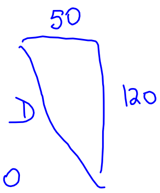
$\frac{dD}{dt} = \frac{1}{D}\left(x \frac{dx}{dt} + y \frac{dy}{dt}\right)$

$= \frac{1}{13\cancel{0}}\left(12\cancel{0}(60) + 5\cancel{0}(25)\right) = \boxed{65\ (mi/h)}$

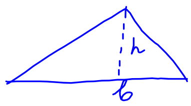
$x = 60(2) = 120$

$y = 25(2) = 50$

$D = \sqrt{120^2 + 50^2} = \sqrt{16900} = 130$



Example 5. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?



h is the altitude
b is the base
A is the area

$\frac{dh}{dt} = 1$

$\frac{dA}{dt} = 2$

$\frac{db}{dt} = ?$ when $h = 10$, $A = 100$

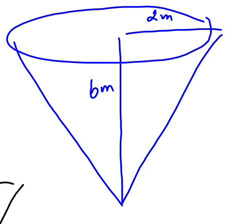
$A = \frac{1}{2} h b \Rightarrow b = \frac{2A}{h}$

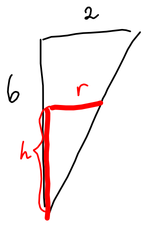
$$\frac{db}{dt} = 2 \cdot \frac{\frac{dA}{dt} \cdot h - \frac{dh}{dt} A}{h^2} = 2 \cdot \frac{2(10) - 1(100)}{10^2} = 2 \cdot \frac{-80}{100} = -1.6$$

the base is decreasing

$10{,}000 \times 10^{-6} = 0.01\ m^3/min$

Example 6. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of $\underbrace{20\ cm/min}_{0.2\ m/min}$ when the height of the water is 2m, find the rate at which water is being pumped into the tank.





h is the height of water at time t

$\frac{dh}{dt} = 0.2$

r is the radius of the cone at the height h.

V is the amount of water at time t.

$\frac{dV}{dt}$ is the rate of change of the amount of water

$\frac{dV}{dt}$ = rate in - rate out.

rate out = 0.01

Let x be the rate at which water is being pumped into the tank.

$\frac{dV}{dt} = x - 0.01$ or $x = \frac{dV}{dt} + 0.01$

Need to find x.

$\boxed{V = \frac{1}{3}\pi r^2 h}$ volume of a cone.

Eliminate r.

similar triangles: $\frac{h}{6} = \frac{r}{2}$ or $r = \frac{h}{3}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{\pi}{27} h^3$

$\frac{dV}{dt} = \frac{\pi}{27} 3h^2 \frac{dh}{dt}$ when $h = 2$

$\frac{dV}{dt} = \frac{\pi}{9} 4 (0.2) = \frac{8\pi}{90}$

$$\boxed{x = \frac{8\pi}{90} + 0.01 \left(m^3/min\right) \approx 0.289}$$

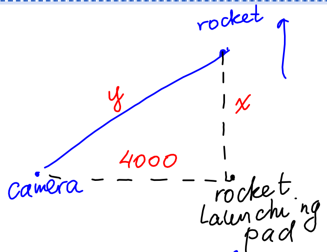
A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 900 ft/s when it has risen 3000 ft. (Round your answers to three decimal places.)

(a) How fast is the distance from the television camera to the rocket changing at that moment?

540 ft/s

(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

0.144 rad/s



x is the height of the rocket

$\frac{dx}{dt} = 900$ (speed of the rocket)

y is the distance between the camera and the rocket.

(a) Find $\frac{dy}{dt}$ when $x = 3000$

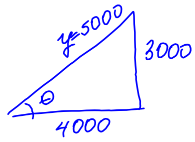
$$\frac{d}{dt}y^2 = \frac{d}{dt}\left(x^2 + (4000)^2\right)$$

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt} + 0$$

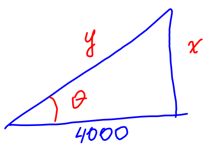
$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{3}{5}(900) = 540 \text{ ft/s}$$

Find y when $x = 3000$



$y^2 = 9\times10^6 + 16\times10^6$
$= 25\times10^6$
$y = 5000$
$\cos\theta = \frac{4}{5}$



(b) Find $\frac{d\theta}{dt}$ when $x = 3000$.

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{x}{4000}$$

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{4000}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{4000}\frac{dx}{dt}$$

$$= \frac{\left(\frac{4}{5}\right)^2}{\cancel{4000}}\cdot\cancel{900}$$

$$= \frac{36}{250} = 0.144$$

$$\frac{d}{dt}\theta = \frac{d}{dt}\arctan\frac{x}{4000}$$

$$\boxed{(\arctan x)' = \frac{1}{1+x^2}}$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x}{4000}\right)^2}\left(\frac{x}{4000}\right)' = \frac{1}{1+\frac{x^2}{16\times10^6}}\cdot\frac{1}{4000}\frac{dx}{dt}$$

$$= \frac{1}{1+\frac{9\times\cancel{10^6}}{16\times\cancel{10^6}}}\cdot\frac{1}{4000}900$$

$$= \frac{1}{1+\frac{9}{16}}\cdot\frac{9}{40} = \frac{1}{\frac{16+9}{16}}\cdot\frac{9}{40}$$

$$= \frac{16}{25}\cdot\frac{9}{40} = \frac{36}{250} = 0.144$$