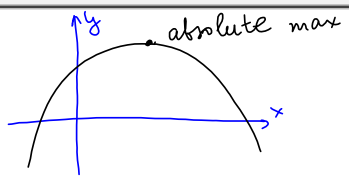
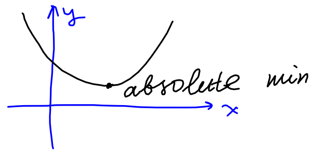
Section 4.1 **Maximum and minimum values**

Definition. Let c be a number in the domain D of a function f. Then $f(c)$ is the

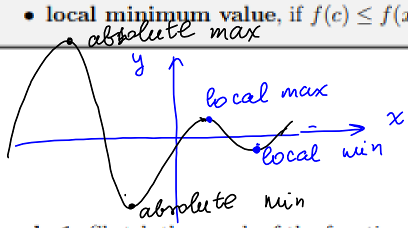
- **absolute maximum value** of f on D, if $f(c) \geq f(x)$ for all x in D.
- **absolute minimum value** of f on D, if $f(c) \leq f(x)$ for all x in D.



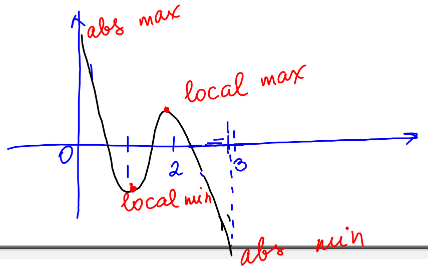


Definition. A number $f(c)$ is a

- **local maximum value**, if $f(c) \geq f(x)$ when x is near c.
- **local minimum value**, if $f(c) \leq f(x)$ when x is near c.



Example 1. Sketch the graph of the function f that is continuous on $[0, 3]$ and has the absolute maximum at 0, absolute minimum at 3, local minimum at 1, local maximum at 2.



The extreme value theorem. If f is continuous on a closed interval $[a, b]$, then f attains ~~an~~ the absolute maximum value $f(c)$ and ~~an~~ the absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$

Definition. A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 2. Find the critical numbers of the function.

(a.) $f(x) = 4x^3 - 9x^2 - 12x + 3$

$f'(x) = 12x^2 - 18x - 12 = 0$

solve for x

$2x^2 - 3x - 2 = 0$

$x_1 = \dfrac{3 + \sqrt{9+16}}{4} = \dfrac{3+5}{4} = 2$

$x_2 = \dfrac{3-5}{4} = -\dfrac{1}{2}$

critical numbers

(b.) $f(x) = \dfrac{x}{x^2+1}$

$f'(x) = \dfrac{(x^2+1)(1) - 2x(x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = 0$

$1 - x^2 = 0$ or

$x^2 + 1 \neq 0$

$\boxed{x = \pm 1 \text{. critical numbers}}$

If f has a local extremum at c, then c is a critical number of f.

The closed interval method. To find the **absolute** maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b)
2. Find $f(a)$ and $f(b)$
3. The largest number of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 3. Find the absolute maximum and absolute minimum values of f on the given interval.

(a.) $f(x) = x^2 + \frac{2}{x}$, $[1/2, 2]$

$$f'(x) = 2x - \frac{2}{x^2} = 0$$

$$\frac{2x^3 - 2}{x^2} = 0$$

$$\frac{2(x^3-1)}{x^2} = 0 \implies x^3 - 1 = 0, \text{ or } x = 1$$

$f'(x)$ DNE if $x^2 = 0$ or $x = 0$ not in $[\frac{1}{2}, 2]$

$$f(1) = 1^2 + \frac{2}{1} = \boxed{3 \text{ abs min}}$$

$$f(\tfrac{1}{2}) = \left(\tfrac{1}{2}\right)^2 + \frac{2}{1/2} = \frac{1}{4} + 4 = \frac{17}{4}$$

$$f(2) = 4 + \frac{2}{2} = \boxed{5 \text{ abs max}}$$

(b.) $f(x) = \cos x + \sin x$, $[0, \pi/3]$

$\left[0, \frac{\pi}{3}\right]$

$f'(x) = -\sin x + \cos x = 0$

$\cos x = \sin x \implies x = \frac{\pi}{4}$

$f(0) = \cos 0 + \sin 0 = \boxed{1 \text{ absolute min}}$

$f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} + \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3}+1}{2} \approx \frac{1.7+1}{2} = \frac{2.7}{2} \approx 1.35$

$f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \approx \boxed{1.4 \text{ absolute max}}$

(c.) $f(x) = xe^{-x}$, $[0, 2]$

$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} = 0$

$e^{-x} \neq 0$, thus $1-x=0$, $x=1$

$f(1) = \boxed{e^{-1} \approx 0.3676 \text{ absolute max}}$

$f(0) = \boxed{0 \text{ absolute min}}$

$f(2) = 2e^{-2} \approx 0.27$

4. 0/0.83 points

Find the critical numbers of the function. (Enter your answers as a comma-separated list. If an answer does not exist, enter DNE.)

$F(x) = x^{4/5}(x - 8)^2$

$x =$ ✗ $0, \frac{16}{7}, 8$

$$f'(x) = \frac{4}{5}x^{-1/5}(x-8)^2 + x^{4/5}2(x-8)$$

$$= \frac{4(x-8)^2}{5x^{1/5}} + 2(x-8)x^{4/5} = 0$$

$$(x-8)\left[\frac{4(x-8)}{5x^{1/5}} + 2x^{4/5}\right] = 0$$

$$(x-8)\frac{4(x-8)+10x}{5x^{1/5}} = 0$$

$x - 8 = 0$

$\boxed{x = 8}$

or

$4x + 10x - 32 = 0$

$14x = 32$

$x = \frac{32}{14} = \boxed{\frac{16}{7} = x}$

$f'(x)$ DNE if $x^{1/5} = 0$ or $\boxed{x = 0}$

Find the absolute maximum and absolute minimum values of f on the given interval.

$f(x) = 4x^3 - 12x^2 - 36x + 3, \quad [-2, 4]$

absolute minimum value -105

absolute maximum value 23

Solution or Explanation

$f'(x) = 12x^2 - 24x - 36 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x_1 = 3, \quad x_2 = -1$

$f(-2) = 4(-2)^3 - 12(-2)^2 - 36(-2) + 3 = -5$

$f(-1) = 4(-1)^3 - 12(-1)^2 - 36(-1) + 3 = 23$ abs max

$f(3) = 4(3)^3 - 12(3^2) - 36(3) + 3 = -105$ abs min

$f(4) = 4(4^3) - 12(4^2) - 36(4) + 3 = -77$

8. 0/0.83 points

Find the absolute minimum and absolute maximum values of f on the given interval.
Hint: You will need to use the identity $\cos 2t = 1 - 2\sin^2 t$ after differentiating in order to be able to factor the derivative.

$f(t) = 12\cos t + 6\sin 2t$, $[0, \pi/2]$

absolute minimum $\boxed{0}$

absolute maximum $\boxed{9\sqrt{3}}$

Tutorial

$f'(t) = -12\sin t + 6\cos 2t\,(2)$

$= \dfrac{-12\sin t + 12\cos 2t}{12} = \dfrac{0}{12}$

$-\sin t + \underbrace{\cos 2t}_{1-2\sin^2 t} = 0$

$-\sin t + 1 - 2\sin^2 t = 0$

$2\sin^2 t + \sin t - 1 = 0$

substitution $\sin t = u$

$2u^2 + u - 1 = 0$

$u_1 = \dfrac{-1+\sqrt{1+8}}{4} = \dfrac{-1+3}{4} = 1/2$

$u_2 = \dfrac{-1-3}{4} = -1$

$\sin t = -1 \Rightarrow t = \frac{3\pi}{2}$ on $[0, \frac{\pi}{2}]$.

$\sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$

$f(0) = 12$

$f(\frac{\pi}{2}) = 0$ abs min

$f(\frac{\pi}{6}) = 12\cos\frac{\pi}{6} + 6\sin\frac{\pi}{3} = 18\cdot\frac{\sqrt{3}}{2} = 9\sqrt{3}$ abs max