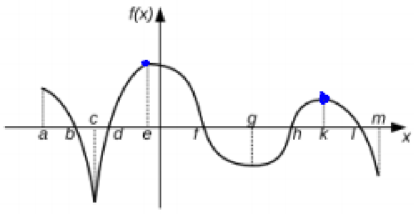
Section 4.3 **How derivatives affect the shape of the graph.**

- If $f'(x) > 0$ on an interval, then f is increasing on that interval
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval
- f has a **local maximum** at the point, where its derivative changes from positive to negative.
- f has a **local minimum** at the point, where its derivative changes from negative to positive.

Example 1. Given the graph of the function f.



(a.) What are the x-coordinate(s) of the points where $f'(x)$ does not exist?

$x = c$

(b.) Identify intervals on which $f'(x) > 0$. f is increasing $(c, e) \cup (g, k)$

$f'(x) < 0$. f is decreasing $(a, c) \cup (e, g) \cup (k, m)$.

(c.) Identify the x coordinates of the points where $f(x)$ has a local maximum.

$x = e$, $x = k$

A local minimum.

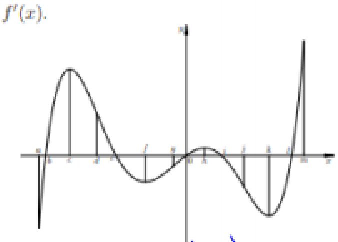
$x = c$, $x = g$

What does f'' say about f?

- If $f''(x) > 0$ on an interval, then f is **concave upward (CU)** on that interval
- If $f''(x) < 0$ on an interval, then f is **concave downward (CD)** on that interval

Definition. A point where curve changes its direction of concavity is called an **inflection point**

Example 2. Given the graph of $f'(x)$.



(a.) Identify intervals on which f is increasing. $(f'(x)>0)$

$(b,e) \cup (0,i) \cup (l,m)$

Is decreasing. $(f'<0)$

$(a,b) \cup (e,0) \cup (i,l)$

(b.) Identify the x coordinates of the points where f has a local maximum.

$x=e,\ x=i$

A local minimum.

$x=b,\ x=0,\ x=l.$

(c.) Identify intervals on which f is concave upward. f' is increasing

$(a,c) \cup (f,h) \cup (k,m)$

Concave downward. f' is decreasing

$(c,f) \cup (h,k)$

(d.) Find the x-coordinates of inflection points.

$x=c,\ x=h,\ x=f,\ x=k$

Example 3. Sketch the graph of a function that satisfies all of the given conditions

Example 3. Sketch the graph of a function that satisfies all of the given conditions.

1. $f'(5) = 0$
2. $f'(x) < 0$ when $x < 5$
3. $f'(x) > 0$ when $x > 5$,
4. $f''(2) = f''(8) = 0$
5. $f''(x) < 0$ when $x < 2$ or $x > 8$
6. $f''(x) > 0$ when $2 < x < 8$
7. $\lim_{x\to\infty} f(x) = 3$ $y=3$ horizontal asymptote when $x\to\infty$
8. $\lim_{x\to-\infty} f(x) = -3$

$y = -3$ horizontal asymptote when $x\to-\infty$.

f': $-$ 5 $+$

$x=5$ is local min for f

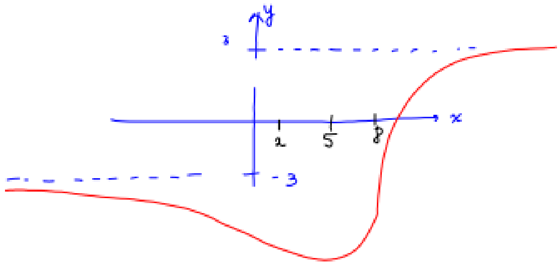
$f''(2) = f''(8) = 0$

f'': $-$ 2 $+$ 8 $-$

f is CU on $(2,8)$

f is CD on $(-\infty,2)\cup(8,\infty)$

$x=2$, $x=8$ inflection points.



The second derivative test. Suppose f'' is continuous near c.

f is CU

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.

f is CD

bf Example 4. For the given functions, find the following:

1. Domain
2. Asymptotes
3. Intercepts
4. Intervals of increase/decrease
5. Local Extrema
6. Intervals of Concavity
7. Inflection Points
8. Sketch a graph

$f(x) = x^4 - 6x^2$

$f(x) = x^4 - 6x^2$

- Domain $-\infty < x < \infty$
- no asymptotes
- intercepts

x - intercepts: $x^4 - 6x^2 = 0$

$x^2(x^2 - 6) = 0$

$x = 0$ or $x = \pm\sqrt{6}$

y - intercept: $y = 0$, $x = 0$.

- $f'(x) = \dfrac{4x^3 - 12x}{4} = \dfrac{0}{4}$

$x^3 - 12x = 0$

$x(x^2 - 3) = 0$

$x = 0$, $x = \pm\sqrt{3}$ critical points.

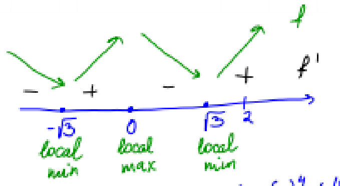
$x(x - \sqrt{3})(x + \sqrt{3}) > 0$

$2(2 - \sqrt{3})(2 + \sqrt{3}) > 0$

$1(1 - \sqrt{3})(1 + \sqrt{3}) < 0$

$-1(-1 - \sqrt{3})(-1 + \sqrt{3}) > 0$

$-2(-2 - \sqrt{3})(-2 + \sqrt{3}) < 0$



f is increasing on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

f is decreasing on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

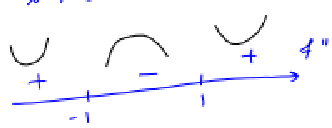
$f(\sqrt{3}) = (\sqrt{3})^4 - 6(\sqrt{3})^2$

$= 9 - 18 = -9$

$f(-\sqrt{3}) = -9$

Concavity. $f''(x) = \dfrac{12x^2 - 12}{12} = \dfrac{0}{12} \Rightarrow x^2 - 1 = 0$ or $x = \pm 1$ inflection points.

$(x-1)(x+1) > 0$



$x = \pm 1$ inflection points

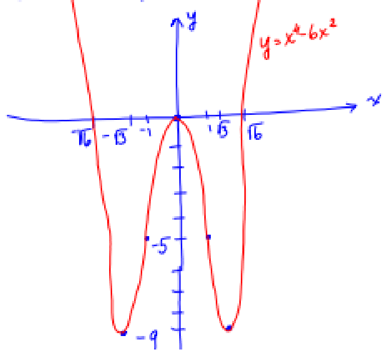
f is CU on $(-\infty, -1) \cup (1, \infty)$

f is CD on $(-1, 1)$

$f(-1) = f(1)$

$= 1 - 6 = -5$

inflection points



$f(x) = \frac{x}{(x-1)^2}$

- Domain $x \neq 1$
- $x=1$ is the vertical asymptote

$\lim_{x\to\infty} \frac{x}{(x-1)^2} = 0$

$\lim_{x\to-\infty} \frac{x}{(x-1)^2} = 0$

$y=0$ is the horizontal asymptote.

- Intercepts: $(0,0)$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{x}{(x-1)^2} = \infty$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{x}{(x-1)^2} = \infty$

$$f'(x) = \left(\frac{x}{(x-1)^2}\right)' = \frac{(x-1)^{\cancel{2}} - 2x\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}}$$

$$= \frac{x-1-2x}{(x-1)^3} = \frac{-1-x}{(x-1)^3} = 0$$

$x=1$ not a critical point (not in the domain)

$-1-x=0$, $x=-1$ - critical point

Intervals of increase/decrease

$-\frac{1+x}{(x-1)^3} > 0$

$-$ $+$ $-$

-1 1

local min

$\boxed{\begin{array}{l} f \text{ is increasing on } (-1,1) \\ f \text{ is decreasing on } (-\infty,-1)\cup(1,\infty) \\ \text{local min @ } (-1,-1/4) \end{array}}$

$f(-1) = \frac{-1}{(-2)^2} = -\frac{1}{4}$

$$f''(x) = \left(-\frac{1+x}{(x-1)^3}\right)' = -\frac{(x-1)^{\cancel{3}} - 3\cancel{(x-1)^2}(1+x)}{(x-1)^{\cancel{6}\,4}} = -\frac{x-1-3-3x}{(x-1)^4} = -\frac{-2x-4}{(x-1)^4} = \frac{2x+4}{(x-1)^4}$$

inflection point $2x+4=0$ or $x=-2$

f is CU on $\frac{2x+4}{(x-1)^4} > 0$

$-$ $+$ $+$

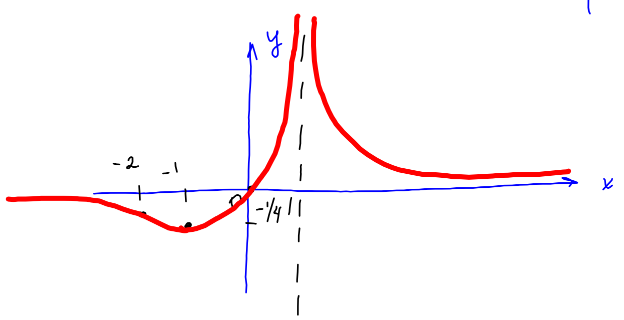
-2 1

f is CU on $(-2,1)\cup(1,\infty)$

f is CD on $(-\infty,-2)$

$f(-2) = \frac{-2}{(-2-1)^2} = -\frac{2}{9}$

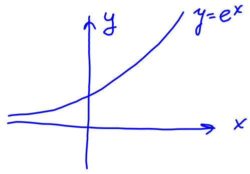
$(-2,-\frac{2}{9})$ inflection point.



$-2/9 < -\frac{1}{4}$

$f(x) = e^{-\frac{1}{x+1}}$

$f(x) = e^{-\frac{1}{x+1}} > 0$



- Domain: $x \neq -1$

  Range: $(0, \infty)$
- no x intercepts, $f(0) = e^{-1} \approx 0.37$
- asymptotes: vertical asymptote $x = -1$

$$\lim_{x\to -1^+} e^{-\frac{1}{x+1}} = \left| \begin{array}{l} y = -\frac{1}{x+1} \\ x \to -1^+ \Rightarrow y \to -\infty \end{array} \right| = \lim_{y \to -\infty} e^y = 0$$

$$\lim_{x\to -1^-} e^{-\frac{1}{x+1}} = \left| \begin{array}{l} y = -\frac{1}{x+1} \\ x \to -1^- \Rightarrow y \to \infty \end{array} \right| = \lim_{y \to \infty} e^y = \infty$$

$$\lim_{x\to\infty} e^{-\frac{1}{x+1}} = e^0 = 1$$

$$\lim_{x\to -\infty} e^{-\frac{1}{x+1}} = e^0 = 1$$

Horizontal asymptotes $y = 1$

- Derivative: $\left(e^{-\frac{1}{x+1}}\right)' = e^{-\frac{1}{x+1}} \cdot \left(-\frac{1}{x+1}\right)' = \dfrac{e^{-\frac{1}{x+1}}}{(x+1)^2} > 0, \quad x \neq -1$

  increases on $(-\infty, -1) \cup (-1, \infty)$

  no local min/max

- 2nd derivative $\left[e^{-\frac{1}{x+1}} (x+1)^{-2}\right]' = e^{-\frac{1}{x+1}} \left(-\frac{1}{x+1}\right)' (x+1)^{-2}$

$$+ e^{-\frac{1}{x+1}} (-2)(x+1)^{-3}$$

$$= e^{-\frac{1}{x+1}} \left[\frac{1}{(x+1)^2} \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right]$$

$$= e^{-\frac{1}{x+1}} \frac{1 - 2(x+1)}{(x+1)^4} = \frac{e^{-\frac{1}{x+1}}}{(x+1)^4} (-2x - 1) = 0$$

$-2x - 1 = 0$ or $x = -\frac{1}{2}$ inflection point

$f\left(-\frac{1}{2}\right) = e^{-\frac{1}{-1/2+1}} = e^{-2} \approx 0.135$

$\left(-\frac{1}{2}, 0.135\right)$ inflection point.

f is CU if $f'' > 0$

$+$	-1	$+$	$-\frac{1}{2}$	$-$

f is CU on $(-\infty, -1) \cup (-1, -\frac{1}{2})$

f is CD on $(-\frac{1}{2}, \infty)$