Section 4.4 **Indeterminate forms and L'Hospitale's rule**

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ for points close to a (except, possibly a). Suppose that
$\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$ or $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$. Then

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right| = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

Example 1. Find the limit.

1. $\lim\limits_{x\to 0} \dfrac{x\cos x - \sin x}{x^3}$

2. $\lim\limits_{x\to 0} \dfrac{e^{5x} - 1 - 5x}{x^2}$

3. $\lim\limits_{x\to \infty} \dfrac{\ln(\ln 6x)}{6x}$

Indeterminate products If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} f(x)g(x) = |\infty \cdot 0| = \lim_{x\to a} \frac{f(x)}{1/g(x)} = \lim_{x\to a} \frac{g(x)}{1/f(x)} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right|$$

and now we can use L'Hospital's Rule.

Example 2. Evaluate

1. $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$

2. $\lim_{x\to -\infty} x^2 e^{2x}$

Indeterminate differences If we have to find $\lim_{x\to a}(f(x)-g(x)) = |\infty - \infty|$ ($\lim_{x\to a} f(x) = \infty, \lim_{x\to a} g(x) = \infty$), then we have to convert the difference into a quotient (by using a common denominator or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right|$ and we can use L'Hospital's Rule.

Example 3. Evaluate $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

Indeterminate powers $\lim_{x\to a}[f(x)]^{g(x)} = |0^0 \text{ or } \infty^0 \text{ or } 1^\infty| = \lim_{x\to a} \mathrm{e}^{g(x)\ln f(x)} = \mathrm{e}^{\lim_{x\to a}[g(x)\ln f(x)]}$. Now let's find

$\lim_{x\to a}[g(x)\ln f(x)] = |0\cdot\infty| = \lim_{x\to a} \frac{\ln f(x)}{\frac{1}{g(x)}} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right| = b$

Then
$\lim_{x\to a}[f(x)]^{g(x)} = \mathrm{e}^b$

Example 4. Evaluate

1. $\lim_{x\to 0}(1+x^2)^{\frac{1}{x}}$.

2. $\lim_{x\to\infty} x^{7/x}$