Section 4.4 **Indeterminate forms and L'Hospital's rule**

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ for points close to a (except, possibly a). Suppose that
$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Then

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right| = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

Example 1. Find the limit.

1. $\lim_{x\to 0}\frac{x\cos x - \sin x}{x^3}$ $= \left|\frac{0}{0}\right| = \lim_{x\to 0}\frac{(x\cos x - \sin x)'}{(x^3)'} = \lim_{x\to 0}\frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{3x^2} = \lim_{x\to 0}\frac{-\cancel{x}\sin x}{3x^{\cancel{2}}}$

$= -\lim_{x\to 0}\frac{\sin x}{3x} = \left|\frac{0}{0}\right| = -\lim_{x\to 0}\frac{(\sin x)'}{(3x)'} = -\lim_{x\to 0}\frac{\cos x}{3} = \boxed{-\frac{1}{3}}$

2. $\lim_{x\to 0}\frac{e^{5x} - 1 - 5x}{x^2}$ $= \left|\frac{0}{0}\right| = \lim_{x\to 0}\frac{(e^{5x} - 1 - 5x)'}{(x^2)'} = \lim_{x\to 0}\frac{e^{5x}(5) - 5}{2x}$ $\left|\frac{0}{0}\right|$

$= \lim_{x\to 0}\frac{(5e^{5x} - 5)'}{(2x)'} = \lim_{x\to 0}\frac{25e^{5x}}{2} = \boxed{\frac{25}{2}}$

3. $\lim_{x\to\infty}\frac{\ln(\ln 6x)}{6x}$ $\left|\frac{\infty}{\infty}\right| = \lim_{x\to\infty}\frac{(\ln(\ln 6x))'}{(6x)'} = \lim_{x\to\infty}\frac{\frac{1}{\ln(6x)}(\ln 6x)'}{6} = \lim_{x\to\infty}\frac{\frac{1}{\ln 6x}\cdot\frac{1}{\cancel{6}x}\cdot\cancel{6}}{6}$

$= \lim_{x\to\infty}\frac{1}{6x\ln 6x} = \frac{1}{\infty} = \boxed{0}$

Indeterminate products If $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$ or $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$, then

$$\lim_{x\to a} f(x)g(x) = |\infty\cdot 0| = \lim_{x\to a} \frac{f(x)}{1/g(x)} = \lim_{x\to a} \frac{g(x)}{1/f(x)} = \left|\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right|$$

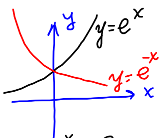
and now we can use L'Hospital's Rule.

Example 2. Evaluate

1. $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$ $\quad |0\cdot\infty| \quad \left[\tan\frac{\pi x}{2} = \frac{1}{\cot\frac{\pi x}{2}}\right] \quad \csc x = \frac{1}{\sin x}$

$$= \lim_{x\to 1}\frac{1-x}{\cot\frac{\pi x}{2}} \left|\frac{0}{0}\right| = \lim_{x\to 1} \frac{(1-x)'}{\left(\cot\frac{\pi x}{2}\right)'} = \lim_{x\to 1} \frac{+1}{+\csc^2\frac{\pi x}{2}\,\frac{\pi}{2}} = \frac{2}{\pi}\lim_{x\to 1} \sin^2\frac{\pi x}{2}$$

$$= \frac{2}{\pi}\sin^2\frac{\pi}{2} = \boxed{\frac{2}{\pi}}$$



$\lim_{x\to-\infty} e^x = 0$

$\lim_{x\to-\infty} e^{-x} = \infty$

2. $\lim_{x\to-\infty} x^2e^{2x}$ $\quad |\infty\cdot 0|$

$$= \lim_{x\to-\infty} \frac{x^2}{e^{-2x}} \left|\frac{\infty}{\infty}\right| = \lim_{x\to-\infty} \frac{(x^2)'}{(e^{-2x})'} = \lim_{x\to-\infty} \frac{\cancel{2}x}{-\cancel{2}e^{-2x}} \left|\frac{-\infty}{\infty}\right|$$

$$= \lim_{x\to-\infty} \frac{(x)'}{(-e^{-2x})'} = \lim_{x\to-\infty} \frac{1}{2e^{-2x}} = \lim_{x\to-\infty} \frac{1}{2}e^{2x} = \boxed{0}$$

Indeterminate differences If we have to find $\lim_{x\to a}(f(x)-g(x)) = |\infty - \infty|$ ($\lim_{x\to a} f(x) = \infty, \lim_{x\to a} g(x) = \infty$), then we have to convert the difference into a quotient (by using a common denominator or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $|\frac{0}{0}$ or $\frac{\infty}{\infty}|$ and we can use L'Hospital's Rule.

Example 3. Evaluate $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$ $|\infty - \infty|$

$$= \lim_{x\to 1} \frac{x\ln x - (x-1)}{(x-1)\ln x} = \lim_{x\to 1} \frac{x\ln x - x + 1}{(x-1)\ln x} = \left|\frac{0}{0}\right|$$

$$= \lim_{x\to 1} \frac{(x\ln x - x + 1)'}{[(x-1)\ln x]'} = \lim_{x\to 1} \frac{\ln x + x\frac{1}{x} - 1}{\ln x + (x-1)\frac{1}{x}} = \lim_{x\to 1} \frac{\ln x}{\ln x + \frac{x-1}{x}}$$

$$= \lim_{x\to 1} \frac{\ln x}{\ln x + 1 - x^{-1}} \quad \left|\frac{0}{0}\right| = \lim_{x\to 1} \frac{(\ln x)'}{(\ln x + 1 - x^{-1})'}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \boxed{\frac{1}{2}}$$

Indeterminate powers $\lim_{x\to a}[f(x)]^{g(x)} = |0^0 \text{ or } \infty^0 \text{ or } 1^\infty| = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a}[g(x)\ln f(x)]}$. Now let's find

$\lim_{x\to a}[g(x)\ln f(x)] = |0\cdot\infty| = \lim_{x\to a}\frac{\ln f(x)}{\frac{1}{g(x)}} = |\frac{0}{0} \text{ or } \frac{\infty}{\infty}| = b$

$$\boxed{[f(x)]^{g(x)} = e^{g(x)\ln f(x)}}$$

Then
$\lim_{x\to a}[f(x)]^{g(x)} = e^b$

Example 4. Evaluate

1. $\lim_{x\to 0}(1+x^2)^{\frac{1}{x}}$. $= \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x^2)} = e^{\lim_{x\to 0}\frac{\ln(1+x^2)}{x}} = e^0 = \boxed{1}$

$$\lim_{x\to 0}\frac{\ln(1+x^2)}{x} \; \left|\frac{0}{0}\right| = \lim_{x\to 0}\frac{(\ln(1+x^2))'}{x'} = \lim_{x\to 0}\frac{\frac{2x}{1+x^2}}{1} = \lim_{x\to 0}\frac{2x}{1+x^2} = 0$$

2. $\lim_{x\to\infty} x^{7/x} = \lim_{x\to\infty} e^{\frac{7}{x}\ln x} = e^{\lim_{x\to\infty}\frac{7\ln x}{x}} = e^0 = \boxed{1}$

$$\lim_{x\to\infty}\frac{7\ln x}{x} \; \left|\frac{\infty}{\infty}\right| = 7\lim_{x\to\infty}\frac{(\ln x)'}{x'} = 7\lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 7\lim_{x\to\infty}\frac{1}{x} = 0$$

3. $\lim_{x\to 0}(1+x)^{\frac{2}{x}} \; |1^\infty| = \lim_{x\to 0} e^{\frac{2}{x}\ln(1+x)} = e^{\lim_{x\to 0}\frac{2\ln(1+x)}{x}} = \boxed{e^2}$

$$\lim_{x\to 0}\frac{2\ln(1+x)}{x} = \left|\frac{0}{0}\right| = 2\lim_{x\to 0}\frac{(\ln(1+x))'}{x'} = 2\lim_{x\to 0}\frac{1}{1+x} = 2$$