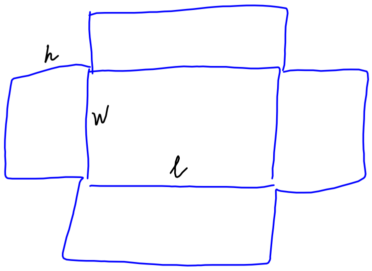
Section 4.7 **Optimization problems.**

Steps in solving applied max and min problems

1. Understand the problem.
2. Draw a diagram.
3. Introduce notation. Assign a symbol to the quantity that is to be minimized or maximized (let us call it Q). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols.
4. Express Q in terms of some of the other symbols from step 3.
5. If Q has been expressed as a function of more than one variable in step 4, use the given information to find relationships (in the form of equation) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus, Q will be given as a function of one variable.
6. Find the **absolute** maximum or minimum of Q.

Example 1. A rectangular storage container with an open top is being built to where the length of the base is three times the width. If there are 240 square feet of material available, find the dimensions of the container that will give the largest volume.



$V = w l h$, $l = 3w$

$V = 3w^2 h$

$S.A. = wl + 2hw + 2lh$

$240 = 3w^2 + 2hw + 6wh$

solve for h:

$8wh = 240 - 3w^2$

$h = \dfrac{240 - 3w^2}{8w}$

updating the volume $\quad V = 3w^{\cancel{2}} \cdot \dfrac{240 - 3w^2}{8\cancel{w}} = \dfrac{3}{8} w(240 - 3w^2)$

$V = 90w - \dfrac{9}{8} w^3$

$V'(w) = 90 - \dfrac{9}{8}(3w^2) = 90 - \dfrac{27}{8} w^2 = 0$

$\dfrac{27}{8} w^2 = 90$

$w^2 = \dfrac{90 \cdot 8}{27} = \dfrac{80}{3}$

$l = \dfrac{\sqrt{3} \cdot 4\sqrt{5}}{\sqrt{3}} = 4\sqrt{3 \cdot 5} = 4\sqrt{15}$

$w = \pm\sqrt{\dfrac{80}{3}} = \pm \dfrac{4\sqrt{5}}{\sqrt{3}}$, $\boxed{w = \dfrac{4\sqrt{5}}{\sqrt{3}}}$ $\boxed{l = 4\sqrt{15}}$ $\boxed{h = \dfrac{240 - 3 \cdot \frac{80}{3}}{8\sqrt{\frac{80}{3}}}}$

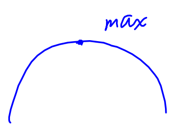
max

$V = 90w - \dfrac{9}{8} w^3$

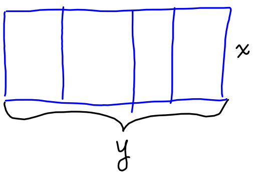
$V' = 90 - \dfrac{27}{8} w^2$

$V'' = -\dfrac{27}{4} w < 0$, if $w > 0$

$V(w)$ is CD on $(0, \infty)$



Example 2. A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of four pens?



area : $A = xy$

$L = 5x + 2y = 750$

$x = \frac{750 - 2y}{5} = 150 - \frac{2}{5}y$

$A = y\left(150 - \frac{2}{5}y\right) = 150y - \frac{2}{5}y^2$

Find the critical points of $A(y) = 150y - \frac{2}{5}y^2$

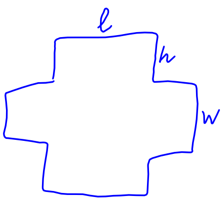
$A'(y) = 150 - \frac{4}{5}y = 0$

$y = \frac{150 \cdot 5}{4} = \frac{375}{2}$ – max

$A''(y) = -\frac{4}{5} < 0 \Rightarrow A(y)$ is CD for all y

$$\boxed{A = 150 \cdot \frac{375}{2} - \frac{2}{5} \cdot \left(\frac{375}{2}\right)^2}$$

Example 3. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost for materials for the cheapest such container.



Cost function

$C = lw(10) + 6(2hw + 2lh)$ minimize

$l = 2w$

update $C = 20w^2 + 12(hw + h(2w))$

$= 20w^2 + 12(3hw)$

Volume $V = lwh$

$= (2w)wh$

$= 2w^2h = 10 \Rightarrow h = \frac{10}{2w^2} = \frac{5}{w^2}$

update $C = 20w^2 + 36\frac{5}{w^2}w$

$C = 20w^2 + \frac{180}{w}$

$C' = \frac{40w - \frac{180}{w^2}}{40} = \frac{0}{40} \Rightarrow w - \frac{9}{2w^2} = 0$ or $w = \frac{9}{2w^2}$ or $w^3 = \frac{9}{2}$

$\boxed{w = \sqrt[3]{\frac{9}{2}}}$

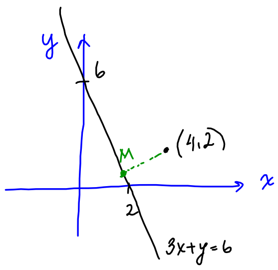
$C'' = 40 + 180(2)w^{-3} > 0$

$C'' = 40 + \frac{360}{w^3} > 0$ for $w > 0$.

C is CU for $w > 0 \Rightarrow w = \sqrt[3]{\frac{9}{2}}$ gives a minimum.

$$\boxed{C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{\frac{9}{2}}}}$$

Example 4. Find the point on the line 3x + y = 6 that is closest to the point (4, 2).



M(x,y)

distance $D=\sqrt{(4-x)^2+(2-y)^2}$ (minimize it)

$3x+y=6,\quad y=6-3x$

$$D=\sqrt{(4-x)^2+(2-(6-3x))^2}$$

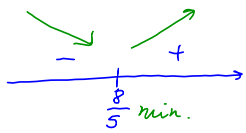
$$=\sqrt{(4-x)^2+(3x-4)^2}$$

$$D'=\frac{(-1)2(4-x)+2(3x-4)(3)}{2\sqrt{(4-x)^2+(3x-4)^2}}=0$$

$$-2(4-x)+6(3x-4)=0$$

$$-8+2x+18x-24=0$$

$$20x=32,\quad x=\frac{32}{20}=\frac{8}{5}$$



$$D'=\frac{(-2)(4-x)+2(3x-4)(3)}{2\sqrt{(4-x)^2+(3x-4)^2}}>0$$

$2x-8+18x-24>0$

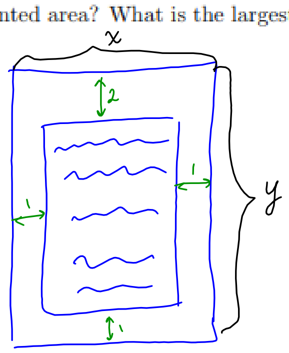
$20x>32$

$x>\frac{32}{20}=\frac{8}{5}$

$$y=6-3\cdot\frac{8}{5}=6-\frac{24}{5}=\frac{6}{5}$$

$$\boxed{\left(\frac{8}{5},\frac{6}{5}\right)}$$

Example 5. A poster is to have a total area of 180 in^2 consisting of a printed area with a 1-inch border at the bottom and sides and a 2-inch border at the top. What dimensions of the whole poster will give the largest printed area? What is the largest printed area?



area of the printed area

$P = (y-3)(x-2)$ — maximize.

$A = xy = 180$

$x = \frac{180}{y}$

$$P = (y-3)\left(\frac{180}{y} - 2\right) = 180 - 2y - \frac{540}{y} + 6$$

$$P = 186 - 2y - \frac{540}{y}$$

$$P' = -2 + \frac{540}{y^2} = 0$$

$$\frac{540}{y^2} = 2 \quad \text{or} \quad y^2 = \frac{540}{2} = 270$$

$$y = \sqrt{270} = \boxed{3\sqrt{30} = y}$$

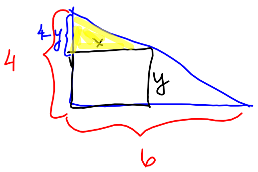
$$P'' = -2(540)\, y^{-3} < 0 \quad \text{for} \quad y > 0.$$

P is CD, $y = 3\sqrt{30}$ gives a maximum.

$$\boxed{x = \frac{180}{3\sqrt{30}} = \frac{60}{\sqrt{30}}}$$

$$\boxed{P = \left(3\sqrt{30} - 3\right)\left(\frac{60}{\sqrt{30}} - 2\right)}$$

Example 6. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 4 cm and 6 cm if two sides of the rectangle lie along the legs.



$A = xy$ ← maximize it.

similar triangles

$\frac{4-y}{4} = \frac{x}{6} \Rightarrow x = \frac{3}{2}(4-y)$

$A = y\,\frac{3}{2}(4-y) = 6y - \frac{3}{2}y^2$

$A' = 6 - 3y = 0 \Rightarrow \boxed{y = 2}$

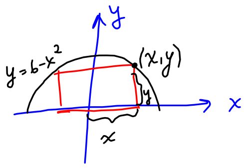
$A'' = -3 < 0$

A is CD, so $y=2$ gives a maximum.

$x = \frac{3}{2}(4-2) = \boxed{3 = x}$

$\boxed{A = 6}$

Example 7. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 6 - x^2$.



$A = 2xy = 2x(6-x^2)$ maximize.

$A = 12x - 2x^3$

$A' = 12 - 6x^2 = 0$

$x^2 = 2, \quad x = \pm\sqrt{2}$

$\boxed{x = \sqrt{2}}$ $(x > 0)$

$A'' = -12x < 0$, if $x > 0$.

A is CD for $x > 0$, $x = \sqrt{2}$ gives a maximum.

$\boxed{y = 6 - x^2 = 6 - 2 = 4}$