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Section 4.9 **Antiderivatives**

**Definition.** Function $F(x)$ is called an bf antiderivative of $f(x)$ on an interval $I$ if

$$\boxed{F'(x) = f(x)}$$

for all $x \in I$.

**Theorem 1.** If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is

$$\boxed{F(x) + C}$$

where $C$ is a constant.

**Table of antidifferentiation formulas**

| Function | Antiderivative |
|---|---|
| $af(x)$, $a$ is a constant | $aF(x) + C$ |
| $f(x) + g(x)$ | $F(x) + G(x) + C$ |
| $a$, $a$ is a constant | $ax + C$ |
| $x$ | $\frac{x^2}{2} + C$ |
| $x^n$, $n \neq -1$ | $\frac{x^{n+1}}{n+1} + C$ |
| $\frac{1}{x} = x^{-1}$ | $\ln\|x\| + C$ |
| $e^x$ | $e^x + C$ |
| $a^x$ | $\frac{a^x}{\ln a} + C$ |
| $\sin x$ | $-\cos x + C$ |
| $\cos x$ | $\sin x + C$ |
| $\sec^2 x$ | $\tan x + C$ |
| $\csc^2 x$ | $-\cot x + C$ |
| $\frac{1}{\sqrt{1-x^2}}$ | $\sin^{-1} x + C$ |
| $\frac{1}{1+x^2}$ | $\tan^{-1} + C$ |

**Example 1.** Find the most general antiderivative of the function.

(a.) $f(x) = x^3 - 4x^2 + 17$

$$F(x) = \frac{x^{3+1}}{3+1} - 4\cdot\frac{x^{2+1}}{2+1} + 17x + C$$

$$= \boxed{\frac{x^4}{4} - 4\frac{x^3}{3} + 17x + C}$$

(b.) $f(t) = \sin(t) - \sqrt{t} = \sin(t) - t^{1/2}$

$$F(t) = -\cos t - \frac{t^{1/2+1}}{1/2+1} + C$$

$$= \boxed{-\cos t - \frac{2t^{3/2}}{3} + C}$$

(c.) $f(x) = (1+x^2)\sqrt[3]{x^2} = (1+x^2)\,x^{2/3} = x^{2/3} + x^2\cdot x^{2/3} = x^{2/3} + x^{8/3}$

$$F(x) = \frac{x^{2/3+1}}{2/3+1} + \frac{x^{8/3+1}}{\frac{8}{3}+1} + C = \boxed{\frac{3}{5}x^{5/3} + \frac{3}{11}x^{11/3} + C}$$

(d.) $f(x) = \dfrac{x^2+x+1}{x} = (x^2+x+1)\,x^{-1} = x + 1 + \frac{1}{x}$

$$F(x) = \boxed{\frac{x^2}{2} + x + \ln|x| + C}$$

(e.) $f(x) = x^e + \dfrac{5}{1+x^2} - \dfrac{1}{\sqrt{1-x^2}}$

$$F(x) = \boxed{\frac{x^{e+1}}{e+1} + 5\arctan x - \arcsin x + C}$$

**Example 2.** Find $f(x)$ if

(a.) $f'(x) = 3\sqrt{x} - \dfrac{1}{\sqrt{x}}$, $f(1) = 2$

$$f'(x) = 3x^{1/2} - x^{-1/2}$$

$$f(x) = 3\frac{x^{1/2+1}}{1/2+1} - \frac{x^{-1/2+1}}{-1/2+1} + C$$

$$= 3\cdot\frac{2}{3}x^{3/2} - 2x^{1/2} + C = 2x^{3/2} - 2x^{1/2} + C$$

$$f(1) = 2\cdot 1^{3/2} - 2\cdot 1^{1/2} + C = 0 + \boxed{C = 2}$$

$$\boxed{f(x) = 2x^{3/2} - 2x^{1/2} + 2}$$

(b.) $f''(x) = x$, $f(0) = -3$, $f'(0) = 2$

$f'(x) = \frac{x^2}{2} + C$ , $f'(0) = \boxed{C = 2}$

$f'(x) = \frac{x^2}{2} + 2$

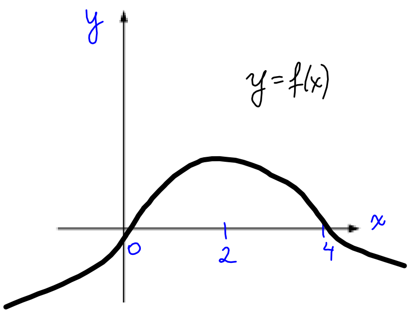
$f(x) = \frac{x^{2+1}}{2(2+1)} + 2x + K = \frac{x^3}{6} + 2x + K$

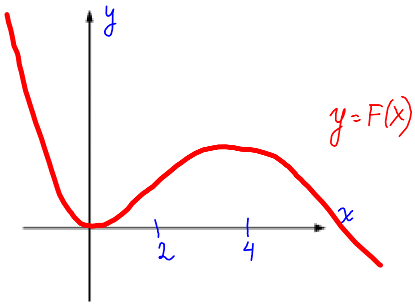
$f(0) = \boxed{K = -3}$

$$\boxed{f(x) = \frac{x^3}{6} + 2x - 3}$$

**The geometry of antiderivatives**

**Example 3.** Given the graph of a function $f(x)$. Make a rough sketch of an antiderivative of $F$, given that $F(0) = 0$.



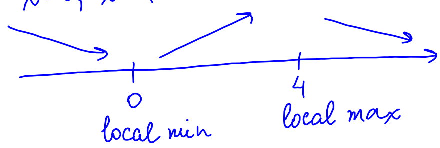

$f(x) > 0$ on $(0, 4)$

$f(x) < 0$ on $(-\infty, 0) \cup (4, \infty)$

$F(x)$ is increasing on $(0, 4)$

$F(x)$ is decreasing on $(-\infty, 0) \cup (4, \infty)$

$x = 0$, $x = 4$ critical values for $F(x)$

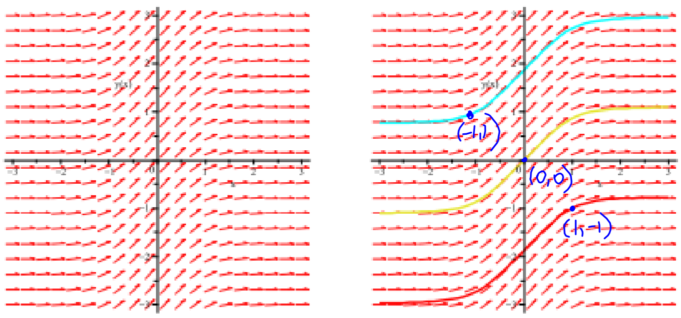


**Example 4.** If

$$f(x) = \frac{1}{x^4 + 1},$$

$F'(x) = f(x)$

sketch the graph of those antiderivatives $F$ that satisfy the initial conditions $F(-1) = 1$, $F(0) = 0$, $F(1) = -1$.



**Rectilinear motion**
If the object has a position function $s = s(t)$, then

$v(t) = s'(t)$ (the position function is an antiderivative for the velocity function),

$a(t) = v'(t)$ (the velocity function is an antiderivative to the acceleration function)

**Example 5.** A particle is moving with the acceleration $a(t) = 3t + 8$, $s(0) = 1$, $v(0) = -2$. Find the position of the particle.

velocity: $v(t) = 3 \cdot \frac{t^2}{2} + 8t + C$

$v(0) = 0 + 0 + \boxed{C = -2}$

$v(t) = \frac{3}{2}t^2 + 8t - 2$

position: $s(t) = \frac{3}{2} \cdot \frac{t^{2+1}}{2+1} + \frac{8t^2}{2} - 2t + K = \frac{1}{2}t^3 + 4t^2 - 2t + K$

$s(0) = \boxed{K = 1}$

$\boxed{s(t) = \frac{1}{2}t^3 + 4t^2 - 2t + 1}$

**Antiderivatives of vector functions**

**Definition.** A vector function $\mathbf{R}(t) =< X(t), Y(t) >$ is called **an antiderivative** of $\mathbf{r}(t) =< x(t), y(t) >$ on an interval $I$ if $\mathbf{R}'(t) = \mathbf{r}(t)$ that is, $X'(t) = x(t)$ and $Y'(t) = y(t)$.

**Theorem 2.** If $\mathbf{R}$ is an antiderivative of $\mathbf{r}$ on an interval $I$, then the most general antiderivative of $\mathbf{r}$ on $I$ is

$$\boxed{\mathbf{R} + \mathbf{C}}$$

where $\mathbf{C}$ is an arbitrary constant vector.

**Example 6.** Find the vector-function that describes the position of a particle that has an acceleration $\mathbf{a}(t) = \cos t\mathbf{i} + e^t\mathbf{j}$ and $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$, $\mathbf{r}(0) = \mathbf{0}$.

$\vec{a}(t) = < \cos t, e^t >$, $\vec{v}(0) = <1, 1>$, $\vec{r}(0) = <0, 0>$

$\vec{v}(t) = < \sin t + C_1, e^t + C_2 >$

$\vec{v}(0) = < C_1, 1 + C_2 > = < 1, 1 > \Rightarrow$ $C_1 = 1$, $1 + C_2 = 1 \Rightarrow C_2 = 0$

$\vec{v}(t) = < \sin t + 1, e^t >$

$\vec{r}(t) = < -\cos t + t + K_1, e^t + K_2 >$

$\vec{r}(0) = < -1 + K_1, 1 + K_2 > = <0, 0> \Rightarrow$ $-1 + K_1 = 0$, $K_1 = 1$; $1 + K_2 = 0$, $K_2 = -1$

$$\boxed{\vec{r}(t) = < -\cos t + t + 1, e^t - 1 >}$$