Section 5.1 **Areas and Distances.**

Sigma notation.

Definition. If a_m, a_{m+1}, a_{m+2},...,a_n are real numbers and m and n are integers such that $m \leq n$, then

$$a_m + a_{m+1} + a_{m+2} + ... + a_n = \sum_{i=m}^{n} a_i$$

Example 1. Write the sum in sigma notation.

1. 1+2+4+8+16+32

2. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$

3. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + ... + \frac{23}{27}$

Example 2. Write the sum in expanded form

1. $\sum_{i=0}^{4} 3^i$

2. $\sum_{i=1}^{n} 2i$, here n is an integer

Theorem. If c is any constant (this means that c does not depend on i), then

1. $\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$
2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
3. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

- $\sum_{i=1}^{n} 1 = n$
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
- $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
- $\sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n-1)}{r-1}$

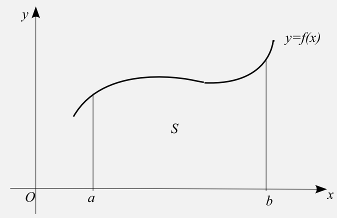
Example 3. Evaluate

1. $\sum_{i=1}^{n} (3+2i)^2$

2. $\sum_{i=1}^{n} (2i+2^i)$

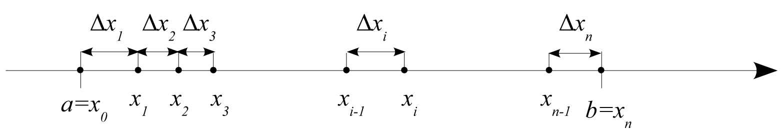
Problem. Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

$$S = \{(x, y) : \; a \leq x \leq b, \; 0 \leq y \leq \; f(x)\}$$



We start by subdividing the interval $[a, b]$ into smaller subintervals by choosing partition points x_0, x_1, x_2,...,x_n so that

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$



Then the n subintervals are

$$[x_0, x_1], [x_1, x_2]...[x_{n-1}, x_n]$$

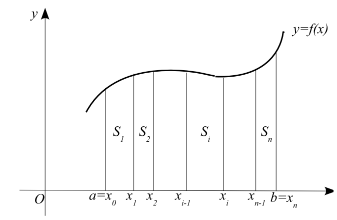
This subdivision is called a **partition** of $[a, b]$ and we denote it by P. We use the notation Δx_i for the length of the ith subinterval $[x_{i-1}, x_i]$.

$$\Delta x_i = x_i - x_{i-1}$$

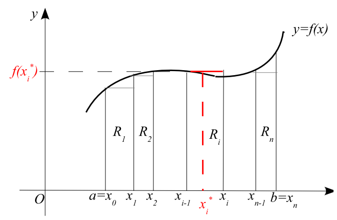
The length of the longest subinterval is denoted by $\|P\|$ and is called the **norm** of P.

$$\|P\| = \max\{\Delta x_1, \Delta x_2, ..., \Delta x_n\}$$

By drawing the lines $x = a$, $x = x_1$, ... $x = b$, we use the partition P to divide the region S into strips S_1, S_2,...,S_n.



We choose a number x_i^* in each subinterval $[x_{i-1}, x_i]$ and construct a rectangle R_i with base Δx_i and height $f(x_i^*)$.



The area of the ith rectangle R_i is

$$A_i = f(x_i^*)\Delta x_i$$

The n rectangles R_1, R_2,...,R_n form a polygonal approximation to the region S.

$$A(S) \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

This approximation becomes better and better as the strips become thinner and thinner, that is, as $\|P\| \to 0$. Then

$$A = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Example 4. Determine a region whose area is equal to

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$$

DO NOT EVALUATE THE LIMIT.

Example 5. Find the area under the curve $y = 1/x^2$ from 1 to 2. Use four subintervals of equal length and take x_i^* to be the midpoint of the ith subinterval.

Example 6. Find the area under the curve $y = x^2 + 3x - 2$ from 1 to 4. Use equal subintervals and take x_i^* to be the right endpoint of the ith subinterval.