Section 5.1 **Areas and Distances.**

Sigma notation.

Definition. If a_m, a_{m+1}, a_{m+2},...,a_n are real numbers and m and n are integers such that $m \leq n$, then

$$a_m + a_{m+1} + a_{m+2} + ... + a_n = \sum_{i=m}^{n} a_i$$

Example 1. Write the sum in sigma notation.

1. 1+2+4+8+16+32 $= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = \boxed{\sum_{i=0}^{5} 2^i}$

2. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} = \boxed{\sum_{i=1}^{5} \frac{1}{i^2}}$

3. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + ... + \frac{23}{27} = \sum_{i=3}^{23} \frac{i}{i+4} = \sum_{j=7}^{27} \frac{j-4}{j}$

Example 2. Write the sum in expanded form

1. $\sum_{i=0}^{4} 3^i$ $= 3^0 + 3^1 + 3^2 + 3^3 + 3^4$
   $= 1 + 3 + 9 + 27 + 81$

2. $\sum_{i=1}^{n} 2i$, here n is an integer
   $= 2(1) + 2(2) + 2(3) + \ldots + 2(n)$

Theorem. If c is any constant (this means that c does not depend on i), then

1. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
3. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

- $\sum_{i=1}^{n} 1 = n$
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
- $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
- $\sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n-1)}{r-1}$

Example 3. Evaluate

1. $\sum_{i=1}^{n} (3+2i)^2 = \sum_{i=1}^{n} (9+6i+4i^2) = \sum_{i=1}^{n} 9 + \sum_{i=1}^{n} 6i + \sum_{i=1}^{n} 4i^2$

$$= 9\underbrace{\sum_{i=1}^{n} 1}_{n} + 6\underbrace{\sum_{i=1}^{n} i}_{\frac{n(n+1)}{2}} + 4\underbrace{\sum_{i=1}^{n} i^2}_{\frac{n(n+1)(2n+1)}{6}}$$

$$= \boxed{9n + 6\frac{n(n+1)}{2} + 4\frac{n(n+1)(2n+1)}{6}}$$

2. $\sum_{i=1}^{n} (2i+2^i) = \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 2^i = = 2\underbrace{\sum_{i=1}^{n} i}_{\frac{n(n+1)}{2}} + \underbrace{\sum_{i=1}^{n} 2^i}_{\frac{2(2^n-1)}{2-1}}$

$$= \boxed{n(n+1) + 2(2^n-1)}$$

$$\sum_{i=1}^{n} i$$

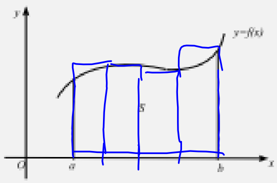
$$\begin{array}{rcccccccccccc} & 1 & + & 2 & + & 3 & + & 4 & + & \dots & + & n \\ + & n & + & (n-1) & + & (n-2) & + & (n-3) & + & \dots & + & 1 \\ \hline \end{array}$$

$$\underbrace{(n+1) + (n+1) + (n+1) + (n+1) + \dots + (n+1)}_{n \text{ terms}} = n(n+1)$$

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

Problem. Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

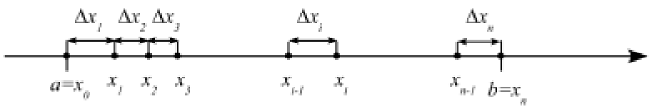
$$S = \{(x, y) : \ a \le x \le b,\ 0 \le y \le \ f(x)\}$$



$A(s) \approx A(\text{rectangle})$

We start by subdividing the interval $[a, b]$ into smaller subintervals by choosing partition points x_0, x_1, x_2,...,x_n so that

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$



Then the n subintervals are

$$[x_0, x_1], [x_1, x_2]...[x_{n-1}, x_n]$$

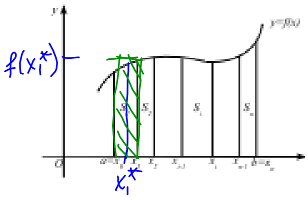
This subdivision is called a **partition** of $[a, b]$ and we denote it by P. We use the notation Δx_i for the length of the ith subinterval $[x_{i-1}, x_i]$.

$$\Delta x_i = x_i - x_{i-1}$$

The length of the longest subinterval is denoted by $\|P\|$ and is called the **norm** of P.

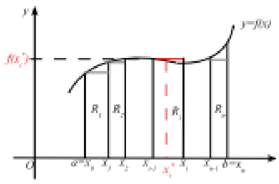
$$\|P\| = \max\{\Delta x_1, \Delta x_2, ..., \Delta x_n\}$$

By drawing the lines $x = a$, $x = x_1$, ... $x = b$, we use the partition P to divide the region S into strips S_1, S_2,...,S_n.



$A(\text{rect}) = \Delta x_1 \cdot f(x_1^*)$

We choose a number x_i^* in each subinterval $[x_{i-1}, x_i]$ and construct a rectangle R_i with base Δx_i and height $f(x_i^*)$.



The area of the ith rectangle R_i is

$$A_i = f(x_i^*)\Delta x_i$$

The n rectangles R_1, R_2,...,R_n form a polygonal approximation to the region S.

$A \approx f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \ldots + f(x_n^*)\Delta x_n$

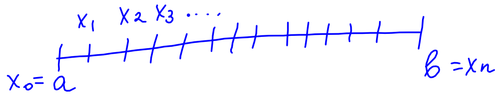
$$A(S) \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

This approximation becomes better and better as the strips become thinner and thinner, that is, as $\|P\| \to 0$. Then

$$\boxed{A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i}$$

If you break $[a,b]$ into n intervals of equal length

$$\Delta x_1 = \Delta x_2 = \Delta x_3 = \ldots = \Delta x_n = \frac{b-a}{n} = \Delta x$$



$$\left\{\begin{aligned} x_1 &= a + \Delta x \\ x_2 &= x_1 + \Delta x \\ &= a + 2\Delta x \\ x_3 &= x_2 + \Delta x \\ &= a + 3\Delta x \\ &\ldots\ldots\ldots \\ x_{n-1} &= a + (n-1)\Delta x \\ x_n &= b \end{aligned}\right.$$

$\|P\| \to 0$ equivalent to $n \to \infty$

Example 4. Determine a region whose area is equal to

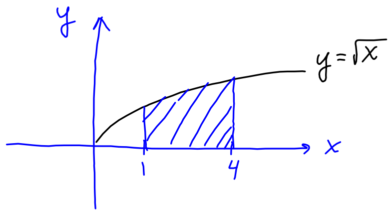
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{3}{n}}_{\Delta x} \underbrace{\sqrt{1+\frac{3i}{n}}}_{f(x_i^*)}$$

DO NOT EVALUATE THE LIMIT.

interval of length 3 $\left(\frac{3}{n} = \frac{b-a}{n} \right)$

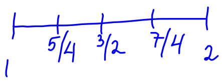
$f(x) = \sqrt{x}$

$x_i^* = \underbrace{1}_{a} + \frac{3}{n} \cdot i$



Example 5. Find the area under the curve $y = 1/x^2$ from 1 to 2. Use four subintervals of equal length and take x_i^* to be the midpoint of the ith subinterval.

$f(x) = \frac{1}{x^2}$, $\quad \Delta x = \frac{1}{4}$



$$\left.\begin{aligned} x_1 &= \frac{5}{4} \\ x_2 &= \frac{3}{2} \\ x_3 &= \frac{7}{4} \\ x_4 &= 2 \end{aligned}\right\} \text{partition points.}$$

midpoints: $[1, \frac{5}{4}]$ $\quad x_1^* = \dfrac{1+\frac{5}{4}}{2} = \dfrac{9}{8}$

$[\frac{5}{4}, \frac{3}{2}]$ $\quad x_2^* = \dfrac{\frac{5}{4}+\frac{3}{2}}{2} = \dfrac{11}{8}$

$[\frac{3}{2}, \frac{7}{4}]$ $\quad x_3^* = \dfrac{\frac{3}{2}+\frac{7}{4}}{2} = \dfrac{13}{8}$

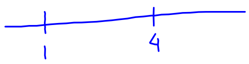
$[\frac{7}{4}, 2]$ $\quad x_4^* = \dfrac{\frac{7}{4}+2}{2} = \dfrac{15}{8}$

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x$$

$$= \frac{1}{4}\left[\frac{1}{(\frac{9}{8})^2} + \frac{1}{(\frac{11}{8})^2} + \frac{1}{(\frac{13}{8})^2} + \frac{1}{(\frac{15}{8})^2}\right]$$

$$= \boxed{\frac{1}{4} \cdot 64\left[\frac{1}{81} + \frac{1}{121} + \frac{1}{169} + \frac{1}{225}\right]}$$

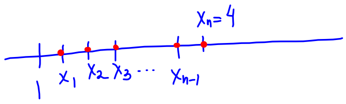
Example 6. Find the area under the curve $y = x^2 + 3x - 2$ from 1 to 4. Use equal subintervals and take x_i^* to be the right endpoint of the ith subinterval.



n subintervals of equal length

$\Delta x = \frac{4-1}{n} = \frac{3}{n}$ - length of subintervals.

Partition points:



$x_0 = 1$

$x_1 = 1 + \Delta x = 1 + \frac{3}{n}$

$x_2 = x_1 + \Delta x = \left(1 + \frac{3}{n}\right) + \frac{3}{n}$

$x_2 = 1 + 2 \cdot \frac{3}{n}$

$x_3 = x_2 + \Delta x = \left(1 + 2\cdot\frac{3}{n}\right) + \frac{3}{n}$

$x_3 = 1 + 3 \cdot \frac{3}{n}$

$$\boxed{x_i = 1 + i \cdot \frac{3}{n}}$$

x_i^* is the right end point of the ith interval

$x_i^* = 1 + i\cdot\frac{3}{n}, \quad i = 1, 2, \dots, n$

$(a+b)^2 = a^2 + 2ab + b^2$

$$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} \Big[\underbrace{\left(1+i\frac{3}{n}\right)^2 + 3\left(1+i\frac{3}{n}\right) - 2}_{(x_i^*)^2 - 3x_i^* - 2}\Big]\underbrace{\frac{3}{n}}_{\Delta x}$$

$$= \sum_{i=1}^{n}\left[\left(1 + 2\cdot\frac{3i}{n} + \left(\frac{3i}{n}\right)^2\right) + 3 + \frac{9i}{n} - 2\right]\frac{3}{n}$$

$$= \sum_{i=1}^{n}\left(2 + \frac{15i}{n} + \frac{9i^2}{n^2}\right)\frac{3}{n} = \sum_{i=1}^{n}\left[\frac{6}{n} + \frac{45i}{n^2} + \frac{27i^2}{n^3}\right]$$

$$= \sum_{i=1}^{n}\frac{6}{n} + \sum_{i=1}^{n}\frac{45i}{n^2} + \sum_{i=1}^{n}\frac{27i^2}{n^3}$$

$$= \frac{6}{n}\underbrace{\sum_{i=1}^{n} 1}_{n} + \frac{45}{n^2}\underbrace{\sum_{i=1}^{n} i}_{\frac{n(n+1)}{2}} + \frac{27}{n^3}\underbrace{\sum_{i=1}^{n} i^2}_{\frac{n(n+1)(2n+1)}{6}}$$

$$= \frac{6}{\cancel{n}}\cdot\cancel{n} + \frac{45}{n^{\cancel{2}}}\cdot\frac{\cancel{n}(n+1)}{2} + \frac{27}{n^{\cancel{3}2}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6}$$

$27\cdot\frac{2}{6} = 9$

$\frac{45}{2}$

$$A = \lim_{n\to\infty}\sum_{n=1}^{\infty} f(x_i^*)\Delta x = \lim_{n\to\infty}\left[6 + \frac{45}{n}\cdot\frac{n+1}{2} + \frac{27}{n^2}\frac{(n+1)(2n+1)}{6}\right]$$

$$= 6 + \frac{45}{2} + 9 = \boxed{\frac{75}{2}}$$